Abstract Wiener groups

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joint with Baudoin, Dobbs, Driver, Eldredge, Gordina

Canonical Wiener space

Let

- $W = W(\mathbb{R}^k) = \{w : [0,1] \to \mathbb{R}^k : w$ is cts and $w(0) = 0\}$ equipped with the sup norm,
- $\mu =$ Wiener measure on W, and
- $H = H(\mathbb{R}^k) =$ Cameron-Martin space, that is,

$$H = \left\{ h \in W : h \text{ is abs cts and } \int_0^1 |\dot{h}(t)|^2 \, dt < \infty \right\}$$

equipped with the inner product

$$\langle h, k \rangle_H := \int_0^1 \dot{h}(t) \cdot \dot{k}(t) \, dt.$$

Canonical Wiener space

Basic facts:

- μ is a Gaussian measure
- The mapping $h \in H \mapsto \dot{h} \in L^2([0,T],\mathbb{R}^k)$ is an isometric isomorphism and H is a separable Hilbert space.
- H is dense in W and $\mu(H) = 0$
- Cameron-Martin-Maruyama quasi-invariance theorem and integration by parts

QI and IBP for Gaussian measure

For $\mu \sim \mathrm{Normal}(0,1)$ on $\mathbb{R}$, we have

$$d\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-|x|^2/2}\, dx.$$

Then, for any $y \in \mathbb{R}$,

$$d\mu^y(x) := d\mu(x-y) = \frac{1}{\sqrt{2\pi}} e^{-|x-y|^2/2}\, dx$$
$$= e^{-|y|^2/2 + \langle x, y \rangle}\, d\mu(x).$$

QI and IBP for Gaussian measure

For $\mu \sim \mathrm{Normal}(0,1)$, for any $y \in \mathbb{R}$,

$$d\mu^y(x) = e^{-|y|^2/2 + \langle x, y \rangle}\, d\mu(x).$$

Thus,

$$\begin{aligned}
\int_{\mathbb{R}} (\partial_y f)(x) d\mu(x) &= \int_{\mathbb{R}} \frac{d}{d\varepsilon}\Big|_0 f(x+\varepsilon y) d\mu(x) \\
&= \frac{d}{d\varepsilon}\Big|_0 \int_{\mathbb{R}} f(x+\varepsilon y) d\mu(x) \\
&= \frac{d}{d\varepsilon}\Big|_0 \int_{\mathbb{R}} f(x) e^{-\varepsilon^2 |y|^2/2 + \langle x, \varepsilon y \rangle}\, d\mu(x) \\
&= \int_{\mathbb{R}} f(x) \frac{d}{d\varepsilon}\Big|_0 e^{-\varepsilon^2 |y|^2/2 + \langle x, \varepsilon y \rangle}\, d\mu(x) \\
&= \int_{\mathbb{R}} f(x) \langle x, y \rangle\, d\mu(x).
\end{aligned}$$

Canonical Wiener space

Theorem (*Cameron-Martin-Maruyama*)
The Wiener measure μ is quasi-invariant under translation by elements of H.

That is, for $y \in H$ and $d\mu^y := d\mu(\cdot - y)$,

$$\mu^y \ll \mu \quad \text{and} \quad \mu^y \gg \mu.$$

More particularly,

$$d\mu^y(x) = e^{-|y|_H^2/2 + \text{``}\langle x,y\rangle\text{''}}\, d\mu(x).$$

Moreover, if $y \notin H$, then $\mu^y \perp \mu$.

Theorem (*Integration by parts*) For all $y \in H$,

$$\int_W (\partial_h f)(x)\, d\mu(x) = \int_W f(x)\, \text{``}\langle x, y\rangle\text{''}\, d\mu(x).$$

Gross' abstract Wiener space

An abstract Wiener space is a triple (W, H, μ) where

- W is a Banach space
- μ is a Gaussian measure on W (for example, $f_*\mu$ is a Gaussian measure on $\mathbb{R}$ for any $f \in W^*$)
- H is a Hilbert space densely embedded in W and, when $\dim(H) = \infty$, $\mu(H) = 0$

The Cameron-Martin-Maruyama Theorem and IBP hold on any abstract Wiener space.

QI and IBP in geometric settings

Theorem (*Shigekawa, 1984*)
Let G be a (fin dim) compact group. Let $W(G)$ be path space on G equipped with "Wiener measure" μ, and let $H(G)$ denote the space of finite-energy paths on G. Then μ is quasi-invariant under translation by elements of $H(G)$ and IBP holds for derivatives in $H(G)$ directions.

other QI and IBP references: Driver (1992), Hsu (1995,2002), Enchev & Stroock (1995), Albeverio, Daletskii, & Kondratiev (1997), Kondratiev, Silva, & Streit (1998), Albeverio, Kondratiev, Röckner, & Tsikalenko (2000), Kuna & Silva (2004), Airault & Malliavin (2006), Driver & Gordina (2008), Hsu & Ouyang (2010),. . .

Smooth measures

A measure μ on $\mathbb{R}^n$ is smooth if

- μ is abs cts wrt Lebesgue measure and the Radon-Nikodym derivative is smooth – that is,

$$\mu = \rho\, dm, \quad \text{for some } \rho \in C^\infty(\mathbb{R}^n, (0, \infty)).$$

$$\Updownarrow$$

- for any multi-index α, there exists $g_\alpha \in C^\infty(\mathbb{R}^n) \cap L^{\infty-}(\mu)$ such that

$$\int_{\mathbb{R}^n} (-D)^\alpha f \, d\mu = \int_{\mathbb{R}^n} f g_\alpha \, d\mu, \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n).$$

Heisenberg group: elliptic case

On $\mathbb{R}^3$, consider the vector fields

$$\left.\begin{aligned}\tilde{X}_1(x) &= \left(1, 0, -\tfrac{1}{2}x_2\right)\\ \tilde{X}_2(x) &= \left(0, 1, \tfrac{1}{2}x_1\right)\\ \tilde{X}_3(x) &= (0, 0, 1)\end{aligned}\right\}$$

NOTE: $\forall x \in \mathbb{R}^3$,

$\mathrm{span}\{\tilde{X}_1(x), \tilde{X}_2(x), \tilde{X}_3(x)\} = \mathbb{R}^3$

Consider the solution $\xi_t = (\xi_t^1, \xi_t^2, \xi_t^3) \in \mathbb{R}^3$ to the SDE

$$\begin{aligned} d\xi_t &= \tilde{X}_1(\xi_t) \circ dB_t^1 + \tilde{X}_2(\xi_t) \circ dB_t^2 + \tilde{X}_3(\xi_t) \circ dB_t^3 \\ &= \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2}\xi_t^2 \end{pmatrix} \circ dB_t^1 + \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2}\xi_t^1 \end{pmatrix} \circ dB_t^2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \circ dB_t^3 \end{aligned}$$

with $\xi_0 = 0$.

Heisenberg group: elliptic case

On $\mathbb{R}^3$, consider the vector fields

$$\tilde{X}_1(x) = \left(1, 0, -\tfrac{1}{2}x_2\right)$$
$$\tilde{X}_2(x) = \left(0, 1, \tfrac{1}{2}x_1\right)$$
$$\tilde{X}_3(x) = (0, 0, 1)$$

The solution to the SDE

$$d\xi_t = \tilde{X}_1(\xi_t) \circ dB_t^1 + \tilde{X}_2(\xi_t) \circ dB_t^2 + \tilde{X}_3(\xi_t) \circ dB_t^3,$$

with $\xi_0 = 0$, may be written explicitly as

$$\xi_t = \left(B_t^1, B_t^2, B_t^3 + \frac{1}{2}\int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1\right)$$

and $\mu_t = \mathrm{Law}(\xi_t)$ is a smooth measure on $\mathbb{R}^3$.

The generator of ξ is $L = \tilde{X}_1^2 + \tilde{X}_2^2 + \tilde{X}_3^2$, which is elliptic.

Heisenberg group: hypoelliptic case

On $\mathbb{R}^3$, consider the vector fields

$$\begin{aligned}
\tilde{X}_1(x) &= \left(1, 0, -\tfrac{1}{2}x_2\right) = \partial_1 - \tfrac{1}{2}x_2\partial_3 \\
\tilde{X}_2(x) &= \left(0, 1, \tfrac{1}{2}x_1\right) = \partial_2 + \tfrac{1}{2}x_1\partial_3 \\
\tilde{X}_3(x) &= (0, 0, 1) = \partial_3
\end{aligned}$$

Note that $[\tilde{X}_1, \tilde{X}_2] := \tilde{X}_1\tilde{X}_2 - \tilde{X}_2\tilde{X}_1 = \tilde{X}_3$. Thus, we can write

$$\mathrm{span}\{\tilde{X}_1(x), \tilde{X}_2(x), [\tilde{X}_1, \tilde{X}_2](x)\} = \mathbb{R}^3.$$

Thus, $\{\tilde{X}_1, \tilde{X}_2\}$ satisfies Hörmander's Condition.

Heisenberg group: hypoelliptic case

On $\mathbb{R}^3$, consider the vector fields

$$\tilde{X}_1(x) = \left(1, 0, -\tfrac{1}{2}x_2\right)$$
$$\tilde{X}_2(x) = \left(0, 1, \tfrac{1}{2}x_1\right)$$
$$\tilde{X}_3(x) = (0, 0, 1)$$

Since $\{\tilde{X}_1, \tilde{X}_2\}$ satisfies Hörmander's Condition, Hörmander's theorem implies that the diffusion satisfying

$$d\eta_t = \tilde{X}_1(\eta_t) \circ dB_t^1 + \tilde{X}_2(\eta_t) \circ dB_t^2,$$

has a smooth measure $\nu_t = \mathrm{Law}(\eta_t)$ on $\mathbb{R}^3$. Again, we may solve this SDE explicitly as

$$\eta_t = \left(B_t^1, B_t^2, \frac{1}{2}\int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1\right).$$

The generator of η is $L = \tilde{X}_1^2 + \tilde{X}_2^2$, which is hypoelliptic.

Heisenberg group geometry

Let $\mathfrak{g} = \operatorname{span}\{X_1, X_2, X_3\} \cong \mathbb{R}^3$ with Lie bracket

$$[X_1, X_2] = X_3, \text{ and all other brackets are } 0.$$

In coordinates, this is

$$[(x_1, x_2, x_3), (x_1', x_2', x_3')] = (0, 0, x_1 x_2' - x_1' x_2).$$

Then via the BCHD formula we may equip $\mathbb{R}^3$ with the group operation

$$\begin{aligned} x \cdot x' &= x + x' + \frac{1}{2}[x, x'] \\ &= \left(x_1 + x_1', x_2 + x_2', x_3 + x_3' + \frac{1}{2}(x_1 x_2' - x_2 x_1') \right). \end{aligned}$$

Then $\mathbb{R}^3$ with this group operation is the Heisenberg group, denoted by G, with $\operatorname{Lie}(G) = \mathfrak{g}$ and

$$\ell_{x*} X_i = \tilde{X}_i(x).$$

Heisenberg group geometry

We can define a left-invariant Riemannian metric on G by taking $\{\tilde{X}_i(x)\}_{i=1}^3$ to be an ONB at each $x \in G$. Then

$$L = \sum_{i=1}^{3} \tilde{X}_i^2$$

is the Laplace-Beltrami operator and

$$d\xi_t = \xi_t \circ dB_t := \ell_{\xi_t *} \circ dB_t$$

$$:= \ell_{\xi_t *} \circ (dB_t^1 X_1 + dB_t^2 X_2 + dB_t^3 X_3) = \sum_{i=1}^{3} \tilde{X}_i(\xi_t) \circ dB_t^i$$

"rolls" the $\mathfrak{g}$-valued BM B_t onto G.

We will call ξ_t Brownian motion on G.

Heisenberg group geometry

Similarly

$$d\eta_t = \tilde{X}_1(\eta_t) \circ dB_t^1 + \tilde{X}_2(\eta_t) \circ dB_t^2$$

rolls $B_t = B_t^1 X_1 + B_t^2 X_2$ a $\mathfrak{g}_0 := \mathrm{span}\{X_1, X_2\}$-valued BM onto G. We will call η_t hypoelliptic Brownian motion on G.

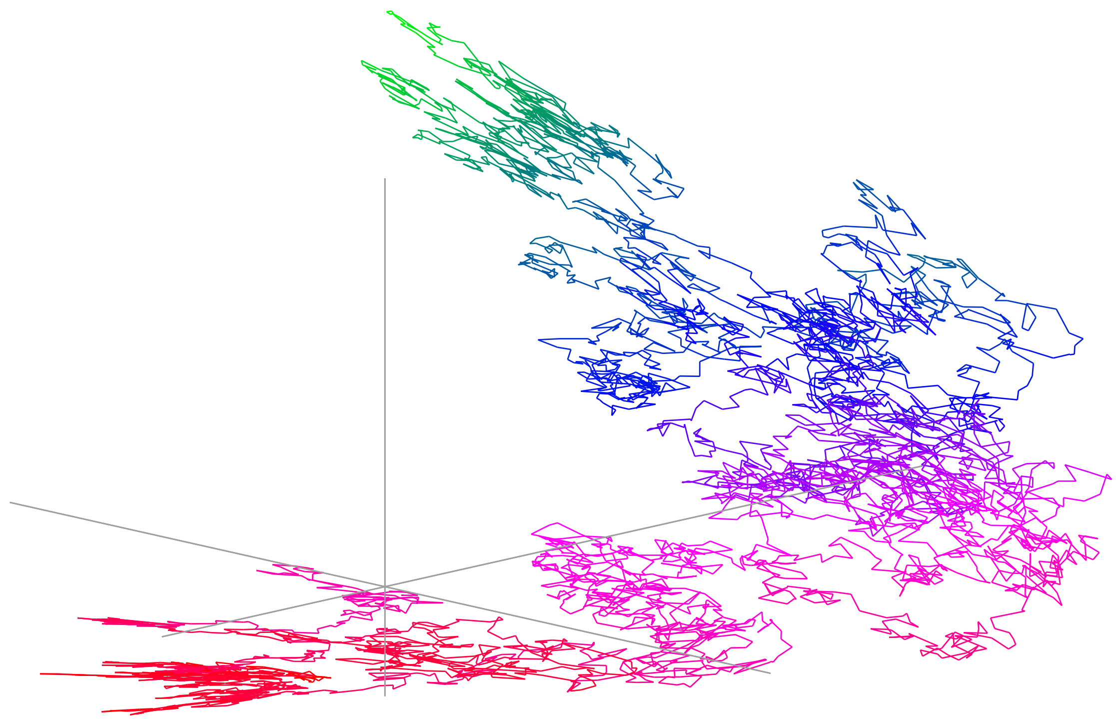
No Riemannian metric! We do have a distance:

$$d_h(x,y) := \inf\{\ell(\gamma) : \gamma \text{ a horizontal path from } x \text{ to } y\}.$$

$$\text{(HC)} \implies d_h(x,y) < \infty \text{ for all } x, y \in G.$$

Hypoelliptic BM on the Heisenberg group

$$\eta_t = \left(B_t^1, B_t^2, \frac{1}{2}\int_0^t B_s^1 dB_s^2 - B_s^2 dB_s^1 \right) = \left(B_t, \frac{1}{2}\int_0^t [B_s, dB_s] \right)$$



(image from Nate Eldredge)

$$\frac{d\nu_t}{dm}(x,y,z) = \frac{1}{16\pi^2}\int_{\mathbb{R}}^3 e^{i\lambda z/2}\frac{\lambda}{\sinh(\lambda t)} e^{-(x^2+y^2)\lambda\coth(\lambda t)/4}\, d\lambda$$

Heat kernel measures on Lie groups

Let G be a Lie group with identity e and Lie algebra $\mathfrak{g}$ with $\dim(\mathfrak{g}) = n$.

Suppose

$$\operatorname{span}(\{X_i\}_{i=1}^n) = \mathfrak{g}$$

and let $\tilde{X}$ denote the unique left invariant v.f. such that $\tilde{X}(e) = X$. Then

$$d\xi_t = \xi_t \circ dB_t := \sum_{i=1}^n \tilde{X}_i(\xi_t) \circ dB_t^i$$

with $\xi_0 = e$, has a smooth law on G.

Heat kernel measures on Lie groups

Let G be a Lie group with identity e and Lie algebra $\mathfrak{g}$ with $\dim(\mathfrak{g}) = n$.

More generally, suppose $\{X_i\}_{i=1}^k \subset \mathfrak{g}$ satisfies

$$\begin{aligned}&\mathrm{Lie}(\{X_i\})\\ &:= \mathrm{span}\{X_i, [X_{i_1}, X_{i_2}], \ldots, [X_{i_1}, [\cdots, [X_{i_{r-1}}, X_{i_r}]]]\} = \mathfrak{g}, \quad \text{(HC)}\end{aligned}$$

and let $\tilde{X}$ denote the unique left invariant v.f. such that $\tilde{X}(e) = X$. Then

$$d\eta_t = \eta_t \circ dB_t := \sum_{i=1}^k \tilde{X}_i(\eta_t) \circ dB_t^i$$

with $\eta_0 = e$, has a smooth law on G – where now

$$B_t = \text{BM on } \mathfrak{g}_0 := \mathrm{span}(\{X_i\}_{i=1}^k) \subsetneq \mathfrak{g}.$$

Heat kernel measures on Lie groups

That is, $\exists\ 0 < p_t \in C^\infty(G)$ such that

$$d\nu_t := \mathrm{Law}(\eta_t) = p_t(\cdot)\, d(Haar).$$

We call p_t the heat kernel and ν_t heat kernel measure.

remark: Note that absolute continuity and positivity immediately imply quasi-invariance:

If $d\nu(x) = p(x)\, d(Haar)(x)$ with $p > 0$, then for any $y \in G$,

$$\begin{aligned} d\nu^y(x) &:= d(\nu \circ r_y^{-1})(x) \\ &= p(xy^{-1})\, d(Haar)(x) = \frac{p(xy^{-1})}{p(x)}\, d\nu(x) \end{aligned}$$

∞-dimensional Heisenberg-like groups

Definition (*Driver and Gordina*) Let (W, H, μ) be an abstract Wiener space and $\mathbf{C}$ be a finite-dimensional inner product space.

Then $\mathfrak{g} = W \times \mathbf{C}$ is a Heisenberg-like Lie algebra if

1. $[W, W] = \mathbf{C}$ and $[W, \mathbf{C}] = [\mathbf{C}, \mathbf{C}] = 0$, and
2. $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathbf{C}$ is continuous.

Let G denote $W \times \mathbf{C}$ when thought of as the associated Lie group with multiplication given by

$$gg' = g + g' + \frac{1}{2}[g, g'].$$

Such a group G will be called an infinite-dimensional Heisenberg-like group.

∞-dimensional Heisenberg-like groups

Let G be an infinite-dimensional Heisenberg-like group with Lie algebra $\mathfrak{g}$.

Definition Let $b_t = (B_t^W, B_t^{\mathbf{C}})$ be BM on $\mathfrak{g}$. Then

$$d\xi_t = \xi_t \circ db_t, \qquad \text{with } \xi_0 = e,$$

is BM on G. This may be solved explicitly as

$$\xi_t = \left(B_t^W, B_t^{\mathbf{C}} + \frac{1}{2}\int_0^t [B_t^W, dB_t^W] \right).$$

Let $\mu_t = \mathrm{Law}(\xi_t)$ denote the heat kernel measure on G.

Example Let $a = (a_j) \in \ell^1(\mathbb{R}^+)$ and set

$$W = \ell_a^2(\mathbb{C}) := \left\{ \{z_j\} \in \mathbb{C}^{\mathbb{N}} : \sum_{j=1}^{\infty} a_j |z_j|^2 < \infty \right\}$$

and $H = \ell^2(\mathbb{C})$. Then (W, H, μ^∞) is an AWS.

For $w = \{w_j\}_{j=1}^\infty = \{x_j + iy_j\}_{j=1}^\infty \in W$ and $c \in \mathbb{R}$,

$$[(w,c),(w',c')] := \left(0, \sum_{j=1}^{\infty} a_j (x_j y_j' - y_j x_j')\right)$$

Via BCHD, define a group operation on $G = W_{\mathrm{Re}} \times \mathbb{R}$ by

$$(w,c) \cdot (w',c') := \left(w + w', c + c' + \frac{1}{2} \sum_{j=1}^{\infty} a_j (x_j y_j' - y_j x_j')\right).$$

∞-dimensional Heisenberg-like groups

Example Let $(W, H) = (\ell_a^2(\mathbb{C}), \ell^2(\mathbb{C}))$ and $\mathbf{C} = \mathbb{R}$.

The solution to

$$d\xi_t = \xi_t \circ db_t, \qquad \text{with } \xi_0 = e,$$

where $b_t = (B_t^W, B_t)$ with $B_t^W = \{X_t^j + iY_t^j\}_{j=1}^{\infty}$, is given by

$$\xi_t = \left(B_t^W, B_t + \frac{1}{2} \sum_{j=1}^{\infty} a_j \int_0^t X_s^j dY_s^j - Y_s^j dX_s^j \right).$$

Elliptic QI theorem on Heisenberg-like groups

Let G be an infinite-dimensional Heisenberg-like group, and ξ_t be BM on G with heat kernel measure $\mu_t = \mathrm{Law}(\xi_t)$.

Let $\mathfrak{g}_{CM}$ denote $H \times \mathbf{C}$ when thought of as a Lie subalgebra of $\mathfrak{g}$, and let G_{CM} denote $H \times \mathbf{C}$ when thought of as a subgroup of G.

Theorem (*Driver and Gordina, 2008*)
For all $y \in G_{CM}$ and $t > 0$, μ_t is quasi-invariant under right translations by y. Moreover, for all $q \in (1, \infty)$,

$$\left\| \frac{d(\mu_t \circ r_y^{-1})}{d\mu_t} \right\|_{L^q(G,\nu_t)} \leq \exp\left(C(k,q,t) d_{CM}(e,y)^2 \right),$$

where $\mathrm{Ric} \geq k$ and d_{CM} is Riemannian distance on G_{CM}.
Similarly for left translations.

Elements of the elliptic proof

Define a nice class of fin-dim projection groups G_P: let P be a nice orthogonal projection into H, and take $G_P = PW \times \mathbf{C}$.

1. Then for ξ^P a Brownian motion on G_P

$$d\xi_t^P = \xi_t^P \circ dPb_t,$$

   we have $\xi^{P_n} \to \xi$ for $P_n|_H \uparrow I_H$.

2. Show that the d_{CM} and $\|\cdot\|_{\mathfrak{g}_{CM}}$ topologies are equivalent, and thus

$$\bigcup_P G_P \text{ is dense in } G_{CM}.$$

3. Show that the $d^{P^n}(x, y) \to d_{CM}(x, y)$ for all $x, y \in G_{P_0} \subset G_{P_n}$ with $P_n|_H \uparrow I_H$.

Elements of the elliptic proof

4. $[\cdot,\cdot] : W \times W \to \mathbf{C}$ cts $\implies$

$$\omega : H \times H \to \mathbf{C} \text{ is Hilbert-Schmidt.}$$

This allows one to prove that $\exists\, k > -\infty$ such that

$$\sup_P \mathrm{Ric}^P \geq k.$$

Then $\mathrm{Ric}^P \geq k \implies$ log Sobolev inequality

$\implies$ Wang/Integrated Harnack inequality:

For all $y \in G_P$ and $q \in (1,\infty)$,

$$\left(\int_{G_P} \left[\frac{p_t^P(xy^{-1})}{p_t^P(x)} \right]^q p_t^P(x)\, dx \right)^{1/q} \leq \exp\left(C(k,q,t) d^P(e,y)^2 \right).$$

Sketch of elliptic proof:

Fix a projection P_0 and let $\{P_n\}_{n=0}^{\infty}$ so that $P_n \uparrow I|_H$. Let $y \in G_0 \subset G_{P_n}$. Then, for any $f \in BC(G)$ and $q \in (1,\infty)$,

$$\int_{G_n} |(f \circ i_n)(xy)|\, d\mu_t^n(x) = \int_{G_n} |(f \circ i_n)(x)| \frac{p_t^n(xy^{-1})}{p_t^n(x)}\, d\mu_t^n(x)$$
$$\leq \|f \circ i_n\|_{L^{q'}(G_n,\mu_t^n)} \exp\left(C(k,q,t) d^n(e,y)^2\right).$$

Taking the limit as $n \to \infty$,

$$\int_G f(xy)\, d\mu_t(x) \leq \|f\|_{L^{q'}(G,\mu_t)} \exp\left(C(k,q,t) d_{CM}(e,y)^2\right).$$

Thus, $d(\mu_t \circ r_y^{-1})/d\mu_t$ exists in L^q for all $q \in (1,\infty)$. ∎

remark: critical estimates based on

$$\mathrm{Ric} \geq -k > -\infty.$$

No lower curvature bounds in hypoelliptic setting.

Problems with $\mathrm{Ric} \geq k$ in hypoelliptic setting

For $f, g \in C^\infty(G)$, let

$$\Gamma(f,g) = \frac{1}{2} L(fg) - fLg - gLf$$
$$\Gamma_2(f) = \frac{1}{2} L\Gamma(f,f) - \Gamma(f,Lf).$$

In particular, if $L = \sum_{i=1}^k \tilde{X}_i^2$, then

$$\Gamma(f,g) = \nabla f \cdot \nabla g = \sum_{i=1}^k (\tilde{X}_i f)(\tilde{X}_i g)$$

$$\Gamma_2(f) = \frac{1}{2} \sum_{i=1}^k L(\tilde{X}_i f)^2 - \sum_{i=1}^k (\tilde{X}_i f)(\tilde{X}_i L f).$$

FACT: $\mathrm{Ric} \geq k \iff$

$$\Gamma_2(f) \geq k\Gamma(f,f), \quad \forall f \in C^\infty(G). \qquad \text{(CDI)}$$

Problems with $\mathrm{Ric} \geq k$ in hypoelliptic setting

Consider the Heisenberg group case again.

$$\tilde{X}(x,y,z) = \partial_x - \frac{1}{2} y \partial_z \qquad \tilde{Y}(x,y,z) = \partial_y + \frac{1}{2} x \partial_z$$

Then $L = \tilde{X}^2 + \tilde{Y}^2$ and

$$\Gamma(f) := \Gamma(f,f) = (\tilde{X} f)^2 + (\tilde{Y} f)^2$$

$$\Gamma_2(f) = \frac{1}{2} L\Gamma(f) - \Gamma(f, Lf)$$

Note that

$$\Gamma(f)(0) = f_x^2 + f_y^2$$

$$\Gamma_2(f)(0) = \sum_{i,j=1}^{2} |\partial_i \partial_j f(0)|^2 + \frac{1}{2} f_z^2(0) + 2(f_y f_{x,z} - f_x f_{y,z})(0).$$

Then there is **no** constant $k \in \mathbb{R}$ so that

$$\Gamma_2(f)(0) \geq k\Gamma(f)(0), \quad \forall f \in C^\infty(G).$$

A replacement for $\mathrm{Ric} \geq k \iff \Gamma_2 \geq k\Gamma$?

Suppose G is a (fin-dim) Lie group with $\mathrm{Lie}(\{X_i\}_{i=1}^k) = \mathfrak{g}$, and let $\{Z_i\}_{i=1}^d$ be an ONB of $\mathrm{span}(\{X_i\}_{i=1}^k)^\perp$. Define

$$\begin{aligned}\Gamma^Z(f,g) &:= \sum_{i=1}^d (\tilde{Z}_i f)(\tilde{Z}_i g) \\ \Gamma_2^Z(f) &:= \frac{1}{2} L\Gamma^Z(f) - \Gamma^Z(f, Lf) \\ &= \frac{1}{2}\sum_{i=1}^N L(\tilde{Z}_i f)^2 - \sum_{i=1}^N (\tilde{Z}_i f)(\tilde{Z}_i Lf).\end{aligned}$$

Suppose there exists $\alpha, \beta > 0$ such that, for all $\lambda > 0$,

$$\Gamma_2(f) + \lambda\Gamma_2^Z(f) \geq \alpha\Gamma^Z(f) - \frac{\beta}{\lambda}\Gamma(f). \qquad \text{(GCDI)}$$

A replacement for $\mathrm{Ric} \geq k \iff \Gamma_2 \geq k\Gamma$

Suppose $\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f))$ and there exists $\alpha, \beta > 0$ such that, for all $\lambda > 0$,

$$\Gamma_2(f) + \lambda \Gamma_2^Z(f) \geq \alpha \Gamma^Z(f) - \frac{\beta}{\lambda}\Gamma(f). \qquad \text{(GCDI)}$$

$\implies$ reverse log Sobolev inequality:

For all $T > 0$ and nice $f : G \to \mathbb{R}$,

$$\Gamma(\ln P_T f) \leq \frac{1 + \frac{2\beta}{\alpha}}{T}\left(\frac{P_T(f \ln f)}{P_T f} - \ln P_T f\right)$$

$\implies$ Wang/Integrated Harnack inequality:

For all $y \in G$ and $q \in (1, \infty)$,

$$\left(\int_G \left[\frac{p_t(xy^{-1})}{p_t(x)}\right]^q p_t(x)\, dx\right)^{1/q} \leq \exp\left(\frac{C(q, \alpha, \beta)}{t} d_h(e, y)^2\right).$$

(GCDI) for 3-dim Heisenberg group

$$
\begin{aligned}
\Gamma(f) &= (Xf)^2 + (Yf)^2 \\
\Gamma^Z(f) &= (Zf)^2 \\
\Gamma_2(f) &= (X^2 f)^2 + (Y^2 f)^2 + \frac{1}{2}((XY + YX)f)^2 \\
&\quad + \frac{1}{2}(Zf)^2 - 2(Xf)(YZf) + 2(Yf)(XZf) \\
\Gamma_2^Z(f) &= \frac{1}{2} L\Gamma^Z(f) - \Gamma^Z(f, Lf) \\
&= \frac{1}{2}(X^2 + Y^2)(Zf)^2 - (Zf) \cdot Z(X^2 f + Y^2 f) \\
&= (XZf)^2 + (YZf)^2 + (Zf)(ZX^2 f + ZY^2 f) \\
&\quad - (Zf)(ZX^2 f + ZY^2 f) \\
&= (XZf)^2 + (YZf)^2
\end{aligned}
$$

(GCDI) for 3-dim Heisenberg group

Note that

$$\Gamma_2(f) \geq \frac{1}{2}(Zf)^2 - 2(Xf)(YZf) + 2(Yf)(XZf)$$

For example, taking $\lambda = 1$

$$\begin{aligned}
&\Gamma_2(f) + \Gamma_2^Z(f) \\
&\quad \geq \frac{1}{2}(Zf)^2 - 2(Xf)(YZf) + 2(Yf)(XZf) + (XZf)^2 + (YZf)^2 \\
&\quad = \frac{1}{2}(Zf)^2 + (Xf - YZf)^2 - (Xf)^2 + (Yf + XZf)^2 - (Yf)^2 \\
&\quad \geq \frac{1}{2}(Zf)^2 - (Xf)^2 - (Yf)^2 \\
&\quad = \frac{1}{2}\Gamma^Z(f) - \Gamma(f)
\end{aligned}$$

So (GCDI) holds with $\alpha = \frac{1}{2}$ and $\beta = 1$.

Hypoelliptic BM on Heisenberg-like groups

Let $\mathfrak{g} = W \times \mathbf{C}$ be an infinite-dimensional Heisenberg-like Lie algebra and assume that

$$[W, W] = \mathbf{C}. \qquad \text{(HC)}$$

Then, for B_t^W BM on W, the "hypoelliptic" BM on G is the solution to

$$d\eta_t = \eta_t \circ dB_t^W, \qquad \text{with } \eta_0 = e.$$

This may be solved explicitly as

$$\eta_t = B_t^W + \frac{1}{2}\int_0^t [B_s^W, dB_s^W] = \left(B_t^W, \frac{1}{2}\int_0^t [B_s^W, dB_s^W]\right).$$

Let $\nu_t = \mathrm{Law}(\eta_t)$ be the "hypoelliptic" heat kernel measure.

Theorem (*Baudoin, Gordina, M*)

For all $y \in G_{CM}$ and $t > 0$, ν_t is quasi-invariant under right translations by y.

Moreover, for all $q \in (1, \infty)$,

$$\left\| \frac{d(\nu_t \circ r_y^{-1})}{d\nu_t} \right\|_{L^q(G,\nu_t)} \leq \exp\left(\frac{C(q, \rho, \|[\cdot,\cdot]\|}{t} d_h(e,y)^2 \right),$$

where $\|[\cdot,\cdot]\|^2$ is the HS norm, $\rho \in (0, \infty)$ is determined by the Lie bracket, and d_h is the horizontal distance on G_{CM}. Similarly for left translations.

Elements of hypoelliptic proof

Take $G_P = PW \times \mathbf{C}$ to be the same nice fin-dim projection groups.

1. For ξ^P a hypoelliptic BM on G_P, we have $\xi^{P_n} \to \xi$ for $P_n|_H \uparrow I_H$.
2. Show that the d_h and $\|\cdot\|_{\mathfrak{g}_{CM}} = \sqrt{\|\cdot\|_H^2 + \|\cdot\|_{\mathbf{C}}}$ topologies are equivalent, and thus
$$\bigcup_P G_P \text{ is dense in } G_{CM}.$$
3. Show that the $d_h^{P_n}(x,y) \to d_h(x,y)$ for all $x, y \in G_{P_0} \subset G_{P_n}$ with $P_n|_H \uparrow I_H$.

N.B.: (2) and (3) depend on $\dim(\mathbf{C}) < \infty$.

Elements of hypoelliptic proof

4. Trivially, $\Gamma_P(f, \Gamma_P^Z(f)) = \Gamma_P^Z(f, \Gamma_P(f))$.

5. We prove that for each P and all $\lambda > 0$, there exists $\rho_P \in (0, \infty)$

$$\Gamma_{2,P}(f) + \lambda \Gamma_{2,P}^Z(f) \geq \rho_P \Gamma_P^Z(f) - \frac{\|[\cdot,\cdot]\|_P^2}{\lambda} \Gamma_P(f).$$

Thus, for all projections P

$$\left(\int_{G_P} \left[\frac{p_t^P(xy^{-1})}{p_t^P(x)} \right]^q p_t^P(x)\, dx \right)^{1/q} \leq \exp\left(\frac{C(q, \rho_P, \|[\cdot,\cdot]\|_P^2)}{t} d_h^P(e,y)^2 \right).$$

Sketch of hypoelliptic proof:

Fix a projection P_0 and let $\{P_n\}_{n=0}^\infty$ so that $P_n \uparrow I|_H$. Let $y \in G_0$. Then, for any $f \in BC(G)$ and $q \in (1,\infty)$,

$$\int_{G_n} |(f \circ i_n)(xy)|\, d\nu_t^n(x) = \int_{G_n} |(f \circ i_n)(x)| \frac{p_t^n(xy^{-1})}{p_t^n(x)}\, d\nu_t^n(x)$$

$$\leq \|f \circ i_n\|_{L^{q'}(G_n,\nu_t^n)} \exp\left(\frac{C(q,\rho_P,\|[\cdot,\cdot]\|_P^2)}{t} d_h^n(e,y)^2 \right).$$

Taking the limit as $n \to \infty$,

$$\int_G f(xy)\, d\nu_t(x)$$

$$\leq \|f\|_{L^{q'}(G,\nu_t)} \exp\left(\frac{C(q,\rho_P,\|[\cdot,\cdot]\|_P^2)}{t} d_h(e,y)^2 \right).$$

Thus, $d(\nu_t \circ r_y^{-1})/d\nu_t$ exists in L^q for all $q \in (1,\infty)$. ■

Hypoelliptic heat kernel

More recently, using techniques more specific to the Heisenberg structure, it's been shown that the hypoelliptic heat kernel measure is smooth in the traditional sense:

Theorem (*Driver-Eldredge-M*)
Let $\nu_t = \mathrm{Law}(\eta_t)$. For each $t > 0$,

$$d\nu_t(x,c) = J_t(x,c)d\gamma_t(x)dc$$

where $\gamma_t = \mathrm{Law}(B_t)$ and dc is Lebesgue measure on $\mathbf{C} = \mathbb{R}^m$.

Moreover, this heat kernel is smooth in the sense that, for any $h_1, \ldots, h_n \in \mathfrak{g}_{CM}$, there exists $\Phi = \Phi(h_1, \ldots, h_m) \in L^{\infty-}(\nu_t)$ so that for all nice $f : G \to \mathbb{R}$

$$\int_G (\tilde{h}_1 \cdots \tilde{h}_n f)(x,c)\, d\nu_t(x,c) = \int_G f(x,c)\Phi(x,c)\, d\nu_t(x,c). \qquad (*)$$

Hypoelliptic heat kernel

For $\lambda \in \mathbf{C}$, define $\Omega_\lambda : H \to H$ by

$$\langle \Omega_\lambda h, k \rangle_H = [h, k] \cdot \lambda.$$

For each $t > 0$, define a random linear transformation $\rho_t(B) : \mathbf{C} \to \mathbf{C}$ by

$$\rho_t(B)\lambda \cdot \lambda := \frac{1}{4} \int_0^t \|\Omega_\lambda B_s\|_H^2 \, ds.$$

$$J_t(x, c) = \mathbb{E}\left[\frac{\exp\left(-\frac{1}{2}\rho_t(B)^{-1} c \cdot c\right)}{\sqrt{\det(2\pi \rho_t(B))}} \middle| B_t = x \right]$$

Hypoelliptic heat kernel

Remarks:

- supersedes previous result, as well as (Dobbs-M) showing smoothness for elliptic hkm (although not path space results)
- still requires $\dim(C) < \infty$
- relies on special structure of step 2 stratified groups
- actual smoothness result (v. smoothness of fin dim projections)

Abstract (nilpotent) Wiener groups

Definition Let $(\mathfrak{g}, \mathfrak{g}_{CM}, \mu)$ be an abstract Wiener space such that $\mathfrak{g}_{CM}$ is equipped with a nilpotent Lie bracket.

Let G_{CM} denote $\mathfrak{g}_{CM}$ when thought of as the Lie group with multiplication defined via BCHD formula.

Assumption: $[\cdot,\cdot] : \mathfrak{g}_{CM} \times \mathfrak{g}_{CM} \to \mathfrak{g}_{CM}$ is Hilbert-Schmidt.

Definition Let $\{B_t\}_{t>0}$ be BM on $\mathfrak{g}$ (elliptic case). Then

$$d\xi_t = \xi_t \circ dB_t, \qquad \text{with } \xi_0 = e,$$

is BM on G. For $t > 0$, let $\mu_t = \mathrm{Law}(\xi_t)$ denote the heat kernel measure on G.

We will call (G, G_{CM}, μ_t) an abstract Wiener group.

Quasi-invariance on abstract Wiener groups

Theorem (M)
For all $y \in G_{CM}$ and $t > 0$, μ_t is quasi-invariant under right translations by y. Moreover, for all $q \in (1, \infty)$,

$$\left\| \frac{d(\mu_t \circ r_y^{-1})}{d\mu_t} \right\|_{L^q(G,\nu_t)} \leq \exp\left(C(t,q,k) d_{CM}^2(e,y) \right)$$

where $\mathrm{Ric} \geq k$. Similarly for left translations.

(Final) Remarks:

- general definition, lots of examples
- natural setting for studying hypoellipticity
- robust method for proving quasi-invariance
- but other hypoelliptic models will need other methods....