Fluctuations of TASEP on a ring in the relaxation time scale

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Overview

Motivation

Consider a particle interacting system in the KPZ universality class such as the asymmetric simple exclusion process.

(a) When the system size is *infinity*, then the one point fluctuations (of the location of a particle or the current at a site) is of order $t^{1/3}$ and usually given by *Tracy-Widom distributions*.

(b) When the system size is *finite*, then the one point fluctuations is of order $t^{1/2}$ and usually given by *Gaussian distribution*.

(c) What happens in the *crossover regime* between KPZ dynamics to Gaussian dynamics?

TASEP on a ring

We consider a particular model: TASEP on a ring. Assume there are N particles which are denoted $x_1, x_2, \cdots, x_N$ on a ring of size L which is denoted $\mathbb{Z}_L$. Each particle has an independent clock which will ring after an exponential waiting time with parameter 1. Once a clock rings, it will be reset. And the corresponding particle moves to the next site ($\bar{i} \in \mathbb{Z}_L \to \overline{i+1} \in \mathbb{Z}_L$) provided it is not occupied.

Two equivalent models:

(1) Periodic TASEP on $\mathbb{Z}$: infinitely many copies of the particles by defining $x_{k+N} = x_k + L$ for all $k \in \mathbb{Z}$.

(2) TASEP on the configuration space

$$\mathfrak{X}_N(L) = \{(x_1, x_2 \cdots, x_N); x_i \in \mathbb{Z}, x_1 < x_2 < \cdots < x_N < x_1 + L\}.$$

For this model, we consider the fluctuations of the location of a tagged particle and the current at a fixed location when $L, N \to \infty$ proportionally and time $t = O(L^{3/2})$, which is called the *relaxation scale*.

Understanding the realxation scale

To understand the relaxation scale, we consider the model with step initial condition and map it to a periodic DLPP model.

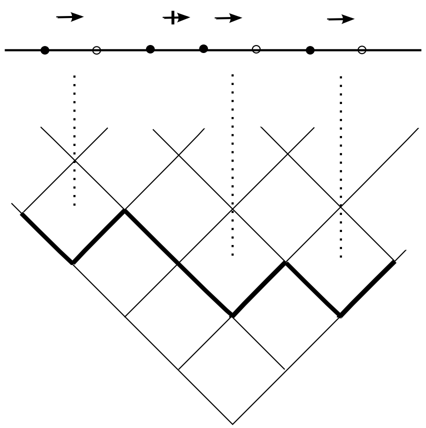

Figure 1 : Mapping TASEP to DLPP

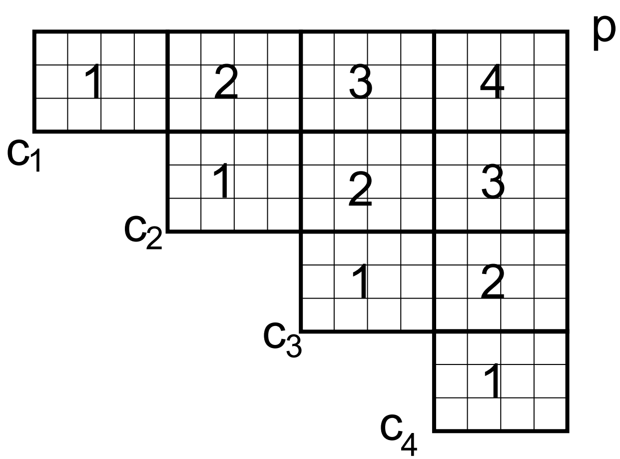

Figure 2 : Periodic DLPP

If $t << L^{3/2}$

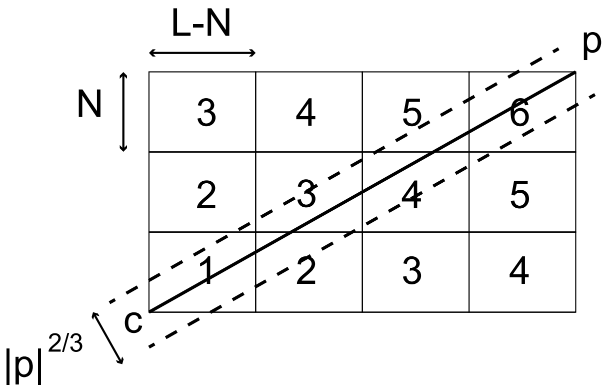

Figure 3 : The maximal paths stay with the dashed lines with high probability.

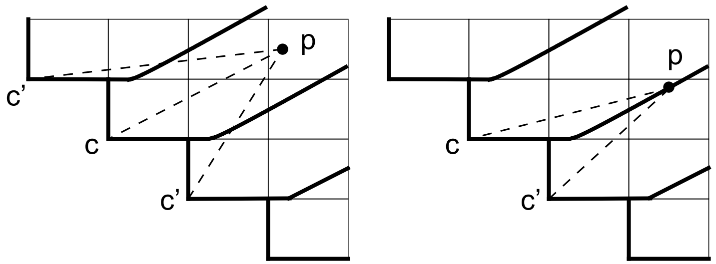

Figure 4 : The limiting distribution is either F_{GUE} or product of two independent F_{GUE}'s, depending on the location of the point.

At the relaxation scale $t = O(L^{3/2})$

The contribution to the maximal path comes from many corners which are correlated.

Known results about this model

(i) Gwa and Spohn [92'] first discussed the relaxation scale.

(ii) Derrida and Lebowitz [98'] obtained the large deviation of the total current in super relaxation scale $t >> L^{3/2}$

(iii) Priezzhev [03'] computed the finite-time transition probability for general initial conditions by adapting the analysis of Schütz [97'] for TASEP on $\mathbb{Z}$.

(iv) Prolhac [15'] computed non-rigorously the limiting distributions for the current fluctuations in the relaxation time scale for the half particle system $L = 2N$ with step/flat/stationary initial conditions. His computation based on two assumptions whose proofs are still missing.

Location of a tagged particle with flat initial condition

Consider the flat initial condition: $x_j(0) = dj$ for $j = 1, 2, \cdots, N$, and $L = dN$, where $d \in \mathbb{Z}_{\geq 2}$ is fixed.

Theorem (Baik-Liu16)

Suppose $\tau > 0$ and $x \in \mathbb{R}$ are both fixed constants. Set

$$t = \frac{\tau}{\sqrt{\rho(1-\rho)}} L^{3/2} \tag{1}$$

then for an arbitrary sequence $k = k_L$ satisfying $1 \leq k_L \leq N$,

$$\lim_{L\to\infty} \mathbb{P}\left(\frac{(x_k(t) - x_k(0)) - (1-\rho)t}{\rho^{-1/3}(1-\rho)^{2/3} t^{1/3}} \geq -x \right) = F_1(\tau^{1/3} x; \tau). \tag{2}$$

Location of a tagged particle with step initial condition

Fix two constants c_1 and c_2 satisfying $0 < c_1 < c_2 < 1$. Let (N_n, L_n) be a sequence of points in

$$B(c_1, c_2) := \{(N, L) \in \mathbb{Z}^2_{\geq 1} : c_1 L \leq N \leq c_2 L\} \tag{3}$$

and satisfy $N_n \to \infty$, $L_n \to \infty$ as $n \to \infty$. Set

$$\rho_n := N_n / L_n \in [c_1, c_2]. \tag{4}$$

Fix $\gamma \in \mathbb{R}$ and let γ_n be a sequence of real numbers satisfying

$$\gamma_n := \gamma + O(L_n^{-1/2}). \tag{5}$$

Set

$$t_n = \frac{L_n}{\rho_n}\left[\frac{\tau\sqrt{\rho_n}}{\sqrt{1-\rho_n}} L_n^{1/2}\right] + \frac{L_n}{\rho_n}\gamma_n + \frac{L_n}{\rho_n}\left(1 - \frac{k_n}{N_n}\right) \tag{6}$$

where $\tau \in \mathbb{R}_{>0}$ is a fixed constant.

Location of a tagged particle with step initial condition

Theorem (Baik-Liu16)

The periodic TASEP associated to the TASEP on a ring of size L_n with the step initial condition $x_j = -N + j$, $1 \leq j \leq N$, satisfies, for an arbitrary sequence of integers k_n satisfying $1 \leq k_n \leq N_n$, and for every fixed $x \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{(x_{k_n}(t_n) - x_{k_n}(0)) - (1-\rho_n)t_n + (1-\rho_n)L_n(1-k_n/N_n)}{\rho_n^{-1/3}(1-\rho_n)^{2/3}t_n^{1/3}} \geq -x \right) \tag{7}$$

converges to $F_2(\tau^{1/3}x; \tau, \gamma)$ as $n \to \infty$.

Current at a tagged location

For the current at a tagged location, we have similar results.

The distribution function $F_1(x;\tau)$

The distribution function for the flat case is defined to be

$$F_1(x;\tau) = \oint e^{xA_1(z)+\tau A_2(z)+A_3(z)+B(z)} \det(I - \mathcal{K}_z^{(1)}) \frac{\mathrm{d}z}{2\pi \mathrm{i} z}, \qquad x \in \mathbb{R}, \tag{8}$$

where the integral is over any simple closed contour in $|z| < 1$ enclosing 0, and

$$A_1(z) := -\frac{1}{\sqrt{2\pi}}\mathrm{Li}_{3/2}(z), \quad A_2(z) := -\frac{1}{\sqrt{2\pi}}\mathrm{Li}_{5/2}(z), \quad A_3(z) := -\frac{1}{4}\log(1-z), \tag{9}$$

and

$$B(z) := \frac{1}{4\pi}\int_0^z \frac{(\mathrm{Li}_{1/2}(y))^2}{y}\mathrm{d}y. \tag{10}$$

The distribution function $F_1(x;\tau)$

The operator $\mathcal{K}_z^{(1)}$ acts on $l^2(S_{z,\mathrm{left}})$ where

$$S_{z,\mathrm{left}} = \{\xi \in \mathbb{C} : e^{-\xi^2/2} = z, \Re\xi < 0\}. \tag{11}$$

The kernel is defined to be

$$\mathcal{K}_z^{(1)}(\xi_1,\xi_2) = \frac{e^{\Psi_z(\xi_1;x,\tau)+\Psi_z(\xi_2;x,\tau)}}{\xi_1(\xi_1+\xi_2)} \tag{12}$$

where

$$\Psi_z(\xi;x,\tau) := -\frac{1}{3}\tau\xi^3 + x\xi - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi} \mathrm{Li}_{1/2}(e^{-w^2/2})\mathrm{d}w. \tag{13}$$

The distribution function $F_1(x;\tau)$

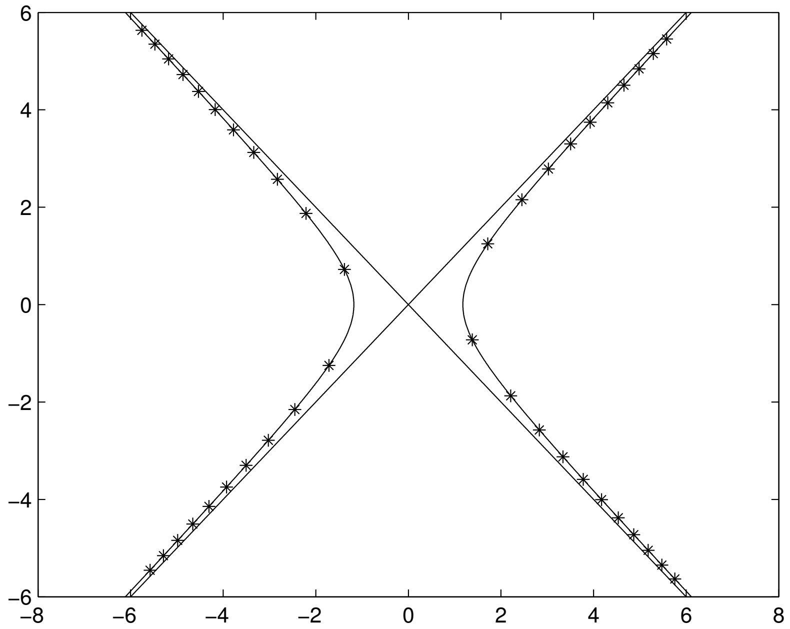

Figure 5 : Illustration of $S_{z,\mathrm{left}}$ ($*$ on the left) with $z = 0.5e^{\mathrm{i}}$.

Properties of $F_1(x;\tau)$

(1) For each $\tau > 0$, $F_1(x;\tau)$ is a distribution function. It is also a continuous function of $\tau > 0$.

(2) For each $x \in \mathbb{R}$, $\lim_{\tau\to 0} F_1(\tau^{1/3}x;\tau) = F_{GOE}(2^{2/3}x)$.

(3) For each $x \in \mathbb{R}$,

$$\lim_{\tau\to\infty} F_1\left(-\tau + \frac{\pi^{1/4}}{\sqrt{2}}x\tau^{1/2};\tau\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy. \qquad (14)$$

[1]

[1]For the large time and small time limit of F_1, we have a formal proof now. A rigorous proof will appear soon. The same for F_2.

Limiting distribution $F_2(x;\tau,\gamma)$

Formula is similar but more complicated.

Properties:

(1) For fixed τ and γ, $F_2(x;\tau,\gamma)$ is a distribution function.

(2) $F_2(x;\tau,\gamma)$ is periodic in γ: $F_2(x;\tau,\gamma+1) = F_2(x;\tau,\gamma)$.

(3) $F_2(x;\tau,\gamma) = F_2(x;\tau,-\gamma)$.

(4) For each $x \in \mathbb{R}$, $\lim_{\tau\to 0} F_2(\tau^{1/3}x;\tau,0) = F_{GUE}(2^{2/3}x)$.

(5) For each $x \in \mathbb{R}$ and $\gamma \in \mathbb{R}$,

$$\lim_{\tau\to\infty} F_2\left(-\tau + \frac{\pi^{1/4}}{\sqrt{2}}x\tau^{1/2};\tau,\gamma\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy. \tag{15}$$

We also expect

$$\lim_{\tau\to 0} F_2\left(\tau^{1/3}x - \frac{\gamma^2}{4\tau};\tau,\gamma\right) = \begin{cases} F_{GUE}(x), & -1/2 < \gamma < 1/2, \\ F_{GUE}(x)^2, & \gamma = 1/2. \end{cases} \tag{16}$$

Transition probability for periodic TASEP

Theorem [J. Baik, Z. Liu]

Denote $P_Y(X;t)$ the transition probability. Then for any $X, Y \in \mathfrak{X}_n(L)$, we have

$$P_Y(X;t) = \oint_0 \det\left[\frac{1}{L}\sum_{z\in R_z} \frac{w^{j-i+1}(w+1)^{-x_i+y_j+i-j}e^{tw}}{w+\rho}\right]_{i,j=1}^N \frac{\mathrm{d}z}{2\pi \mathrm{i} z},$$

where R_z is the set of all roots of $w^N(w+1)^{L-N} = z^L$.

We followed the idea of Tracy-Widom on the formula of ASEP. We consider a system of equations of $u(X;t)$ where $(X;t) = (x_1, \cdots, x_N; t) \in \mathbb{Z}^N \times \mathbb{R}_{\geq 0}$.

$$\text{Master Equations}: \quad \frac{d}{dt}u(X;t) = \sum_{i=1}^{N}(u(X_i;t) - u(X;t))$$

where $X_i := (x_1, \cdots, x_{i-1}, x_i - 1, x_{i+1}, \cdots, x_N)$.

$$\text{Boundary Conditions 1}: \quad u(X_i;t) = u(X;t) \text{ if } x_i = x_{i-1} + 1$$

$$\text{Boundary Conditions 2}: \quad u(X_1;t) = u(X;t) \text{ if } x_N = x_1 + L - 1.$$

$$\text{Initial Condition}: \quad u(X;0) = \delta_Y(X), \text{ for all } X \in \mathfrak{X}_N(L).$$

Without the second boundary condition, it gives the transition probability for TASEP on the integer lattice:

$$\det\left[\oint_{|\xi|=\epsilon}(1-\xi)^{j-i}\xi^{x_i-y_j}e^{t(\xi^{-1}-1)}\frac{d\xi}{2\pi i\xi}\right]_{i,j=1}^{N}.$$

The extra boundary condition gives rise to the discreteness of the sum. The solution is constructive.

Fredholm determinant representation

Let $R_{z,L}, R_{z,R}$ be the left and right parts of the roots set R_z. Define $q_{z,\mathrm{left}}(w) = \prod_{u \in R_{z,\mathrm{left}}}(w-u)$ and $q_{z,\mathrm{right}}(w) = \prod_{v \in R_{z,\mathrm{right}}}(w-v)$.
For the flat initial condition, we have

$$P(x_k \geq a; t) = \oint C_N^{(1)}(z) \det\left(1 + K_z^{(1)}\right) \frac{\mathrm{d}z}{2\pi \mathrm{i} z},$$

where $C_N^{(1)}(z)$ is a function in z and and $K_z^{(1)}$ is acting on $\ell^2(R_{z,\mathrm{left}})$ by the kernel

$$K_z^{(1)}(u,u') = \frac{f_1(u)}{(u-v')f_1(v')}. \tag{17}$$

Here v' is in $R_{z,\mathrm{right}}$ determined by $v'(v'+1)^{d-1} = u'(u'+1)^{d-1}$, and f_1 is some function.

Remarks:

If we let $L \to \infty$ but fix all other parameters, we obtain the Fredholm determinant formula for the one point distribution of TASEP on $\mathbb{Z}$ with flat initial condition. This formula matches the one obtained by Borodin-Ferrari-Prähofer-Sasamoto for $d = 2$.

For step initial condition, we have a similar formula.

The asymptotic analysis is complicated but straightforward.

Thank you!