Large deviations for certain inhomogeneous corner growth models

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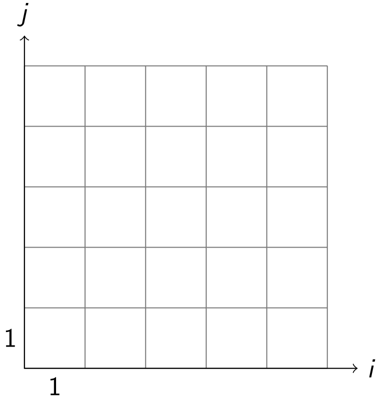
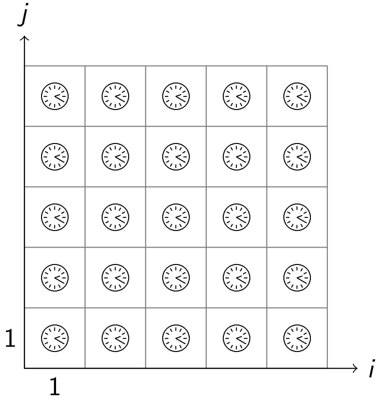
(joint work with Elnur Emrah)

Corner growth model



Corner growth model



Take $W(i,j)$, $(i,j) \in \mathbb{N}^2$.

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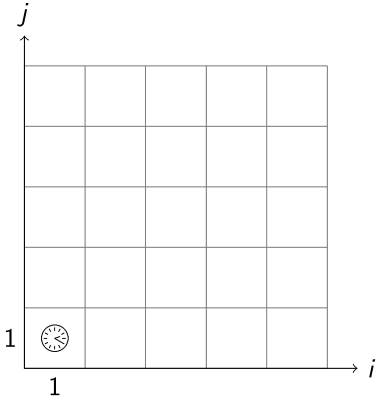

Take $W(i,j)$, $(i,j) \in \mathbb{N}^2$. (m,n) enters at time $G(m,n)$:

$$G(m,n) = G(m-1,n) \vee G(m,n-1) + W(m,n)$$
$$G(m,0) = G(0,n) = 0$$

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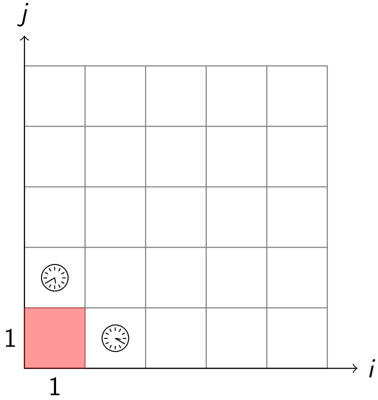

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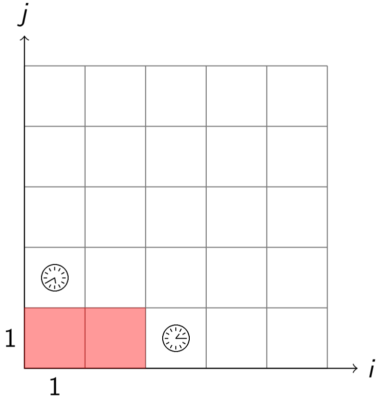

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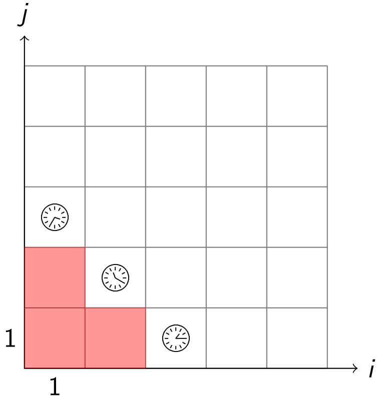

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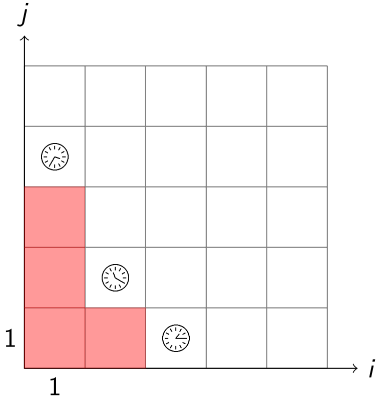

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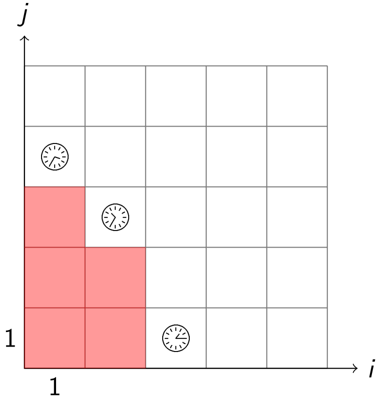

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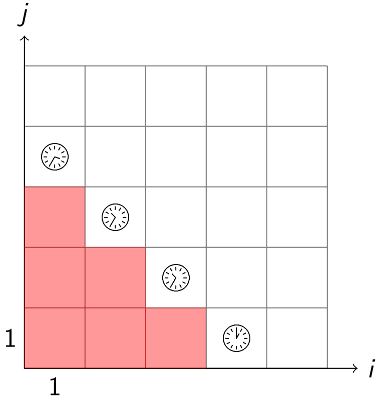

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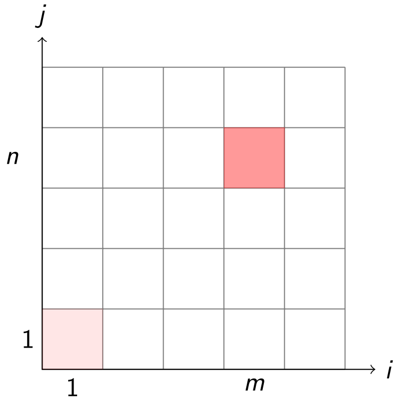
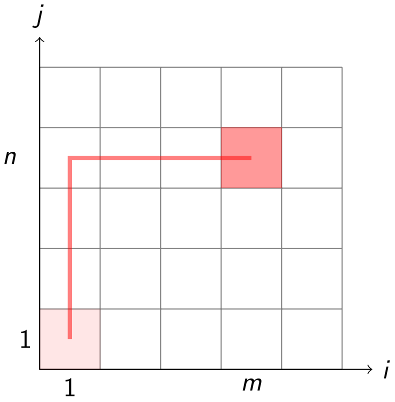
Corner growth model - LPP



By induction,

$$G(m,n) = \max_{\substack{\text{up-right paths} \\ \pi:(1,1)\to(m,n)}} \sum_{(i,j)\in\pi} W(i,j).$$



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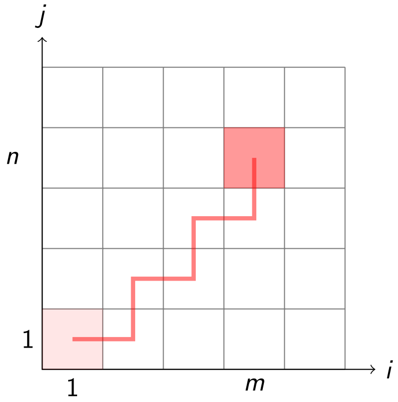

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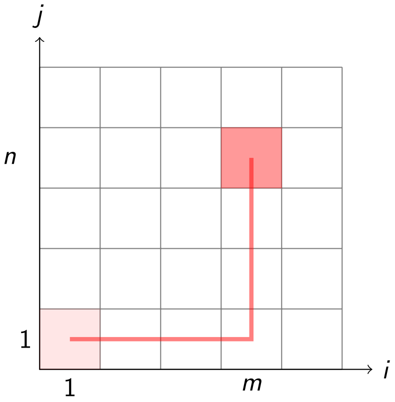

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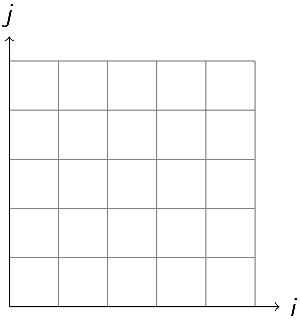
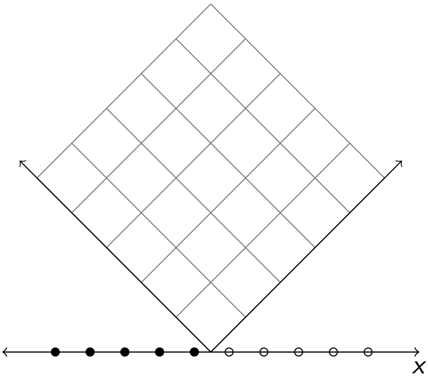
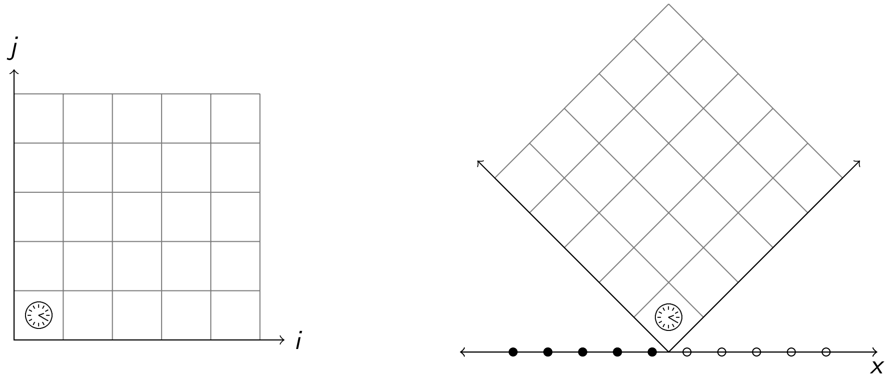




(m, n) enters when particle n (count right-to-left) interchanges with hole m (count left-to-right).

Corner growth model - exclusion



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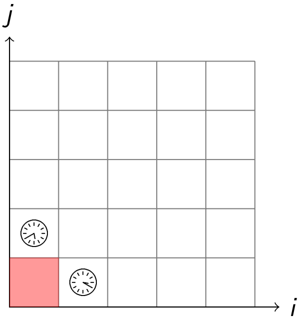

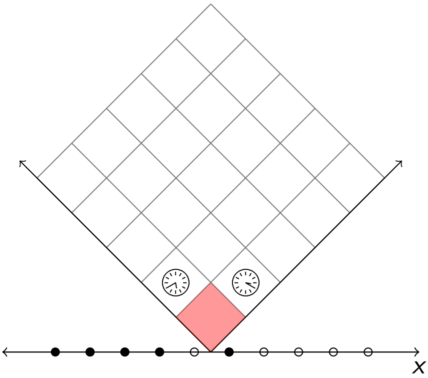

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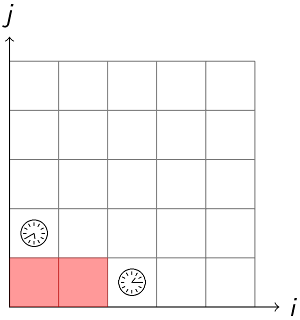

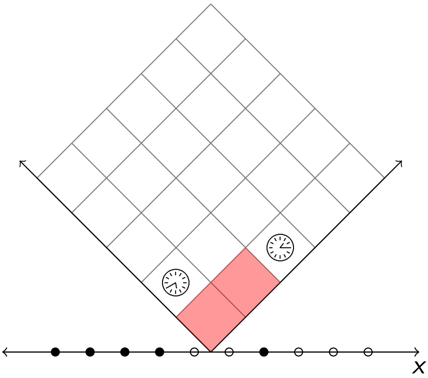

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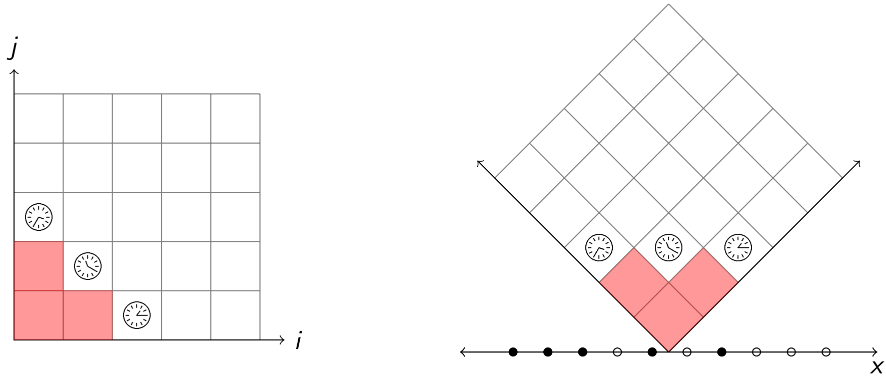

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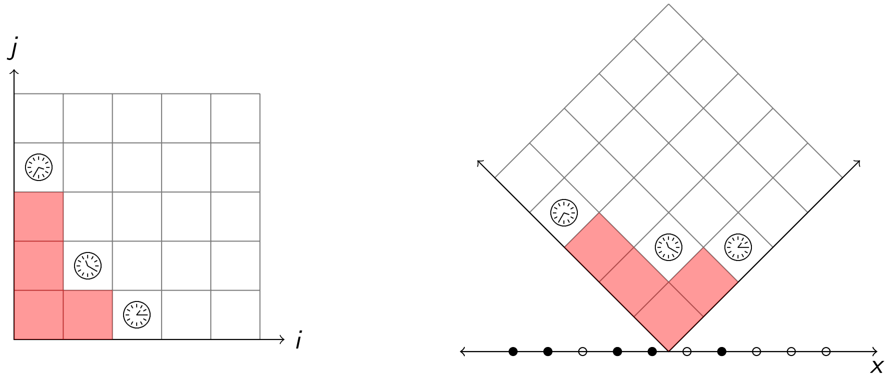

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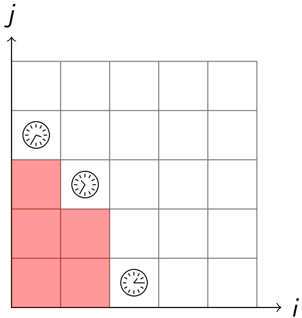

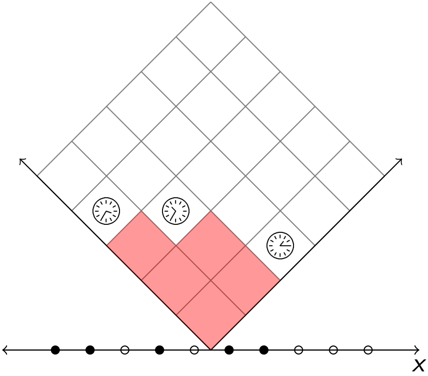

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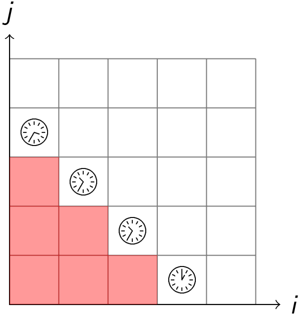

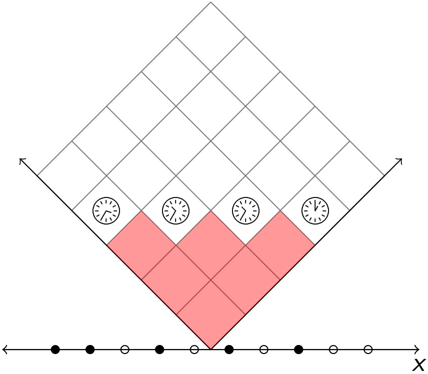

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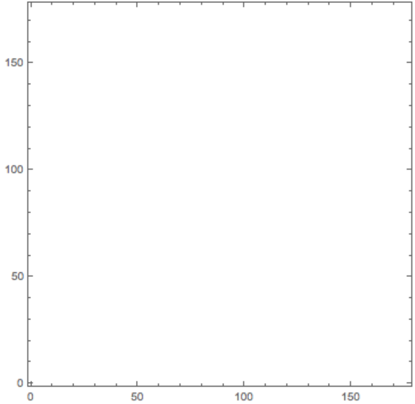
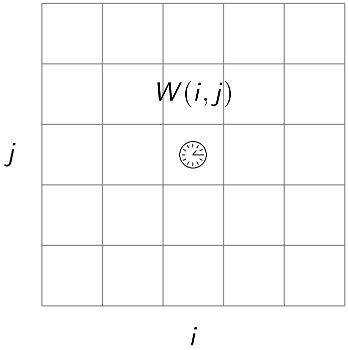
Shape theorem

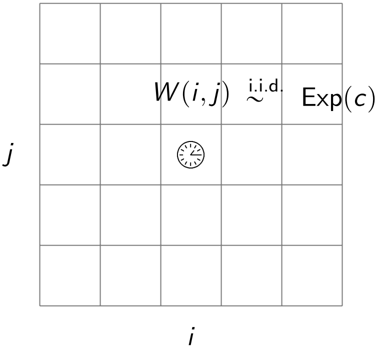
Under mild assumptions,

$$\lim_{n\to\infty} n^{-1} G(\lfloor ns \rfloor, \lfloor nt \rfloor) = g(s,t) \quad \text{(a.s.)}$$

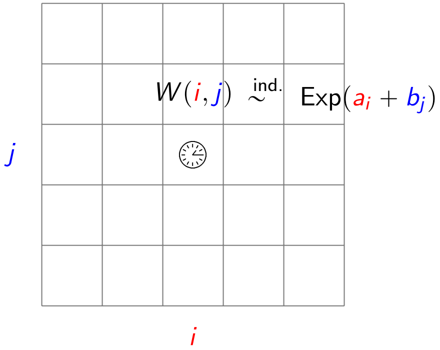


Solvable corner growth models





Homogeneous:
$W(i,j) \overset{i.i.d.}{\sim} \text{Exp}(c)$
(or $\text{Geo}(p)$).



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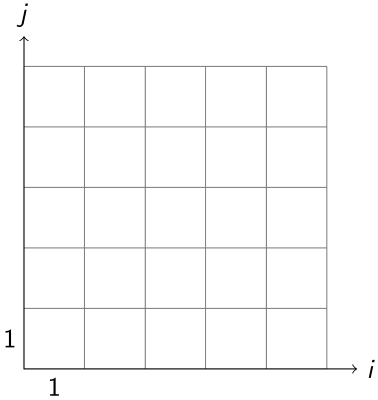
Inhomogeneous:
$W(i,j) \overset{ind.}{\sim} \text{Exp}(a_i + b_j)$

$\mathbf{a} = (a_n)_{n\geqslant 1}$ and $\mathbf{b} = (b_n)_{n\geqslant 1}$
$a_i, b_j \geqslant c > 0$.

Stationary model

Extend environment to include $(i,0),(0,j)$, $i,j \geqslant 0$, $W(0,0)=0$.



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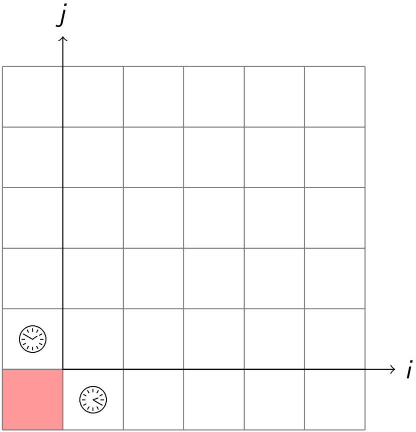

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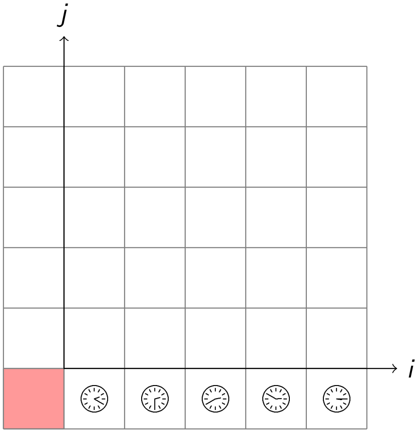

Take z: $-a_i < z < b_j$, $i, j \in \mathbb{N}$

$$W(i, 0) \sim \text{Exp}(a_i + z)$$

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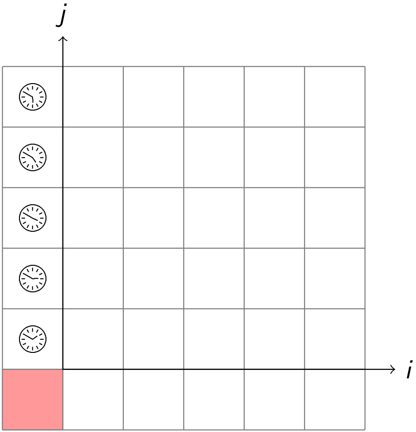

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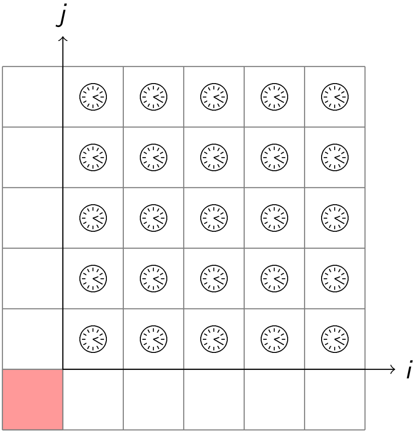

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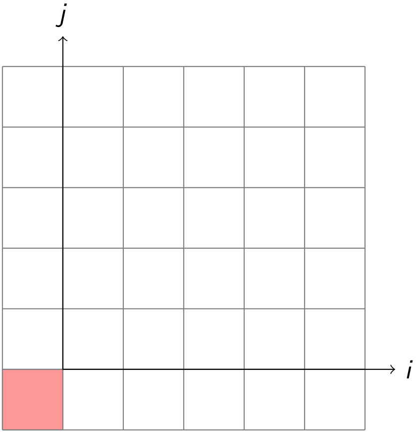

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(i, j) enters at time $\hat{G}(i, j)$

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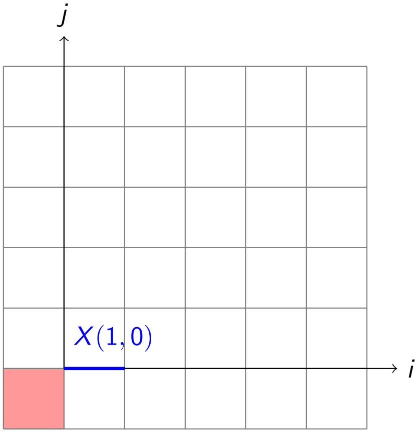

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$$X(i,j) = \hat{G}(i,j) - \hat{G}(i-1,j)$$

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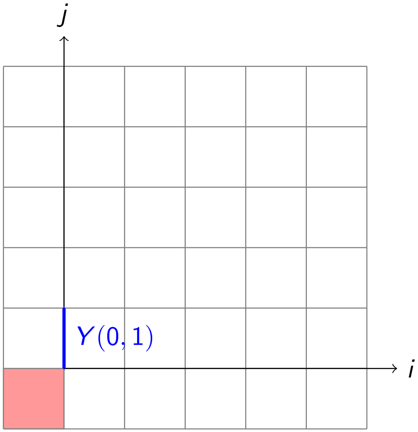

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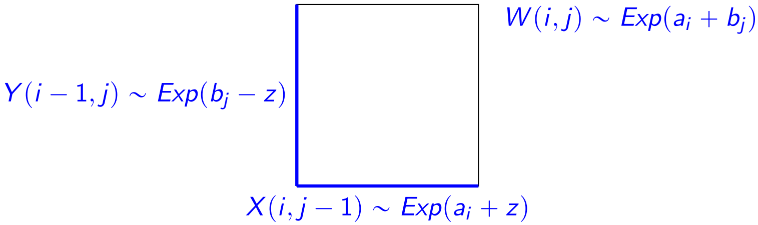
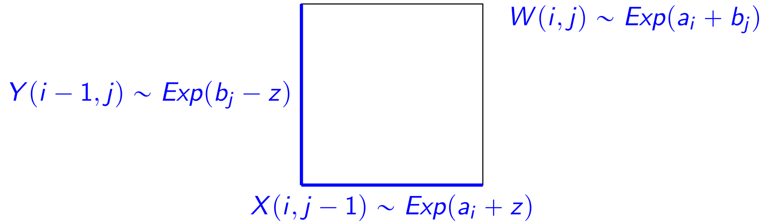
$$Y(i,j) = \hat{G}(i,j) - \hat{G}(i,j-1)$$


$W(i,j) \sim Exp(a_i + b_j)$
$Y(i-1,j) \sim Exp(b_j - z)$
$X(i,j-1) \sim Exp(a_i + z)$

Stationary model



$$X(i,j) = X(i,j-1) - X(i,j-1) \wedge Y(i-1,j) + W(i,j)$$
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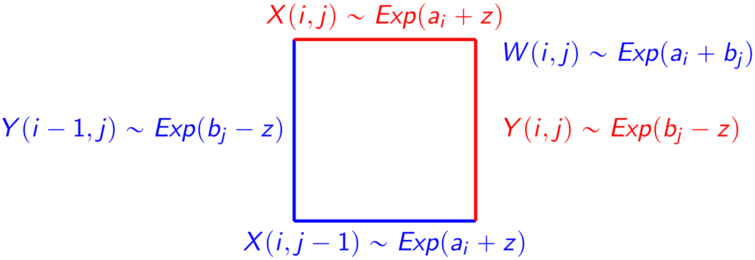

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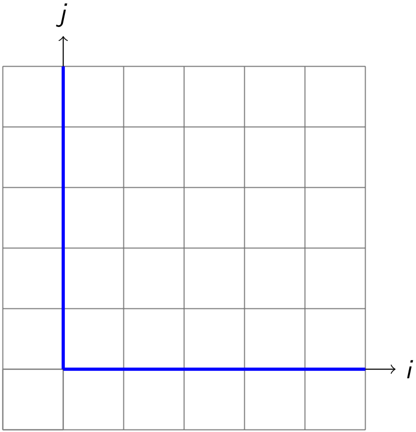
Key point: This map preserves the joint distribution of (X, Y).

Stationary model



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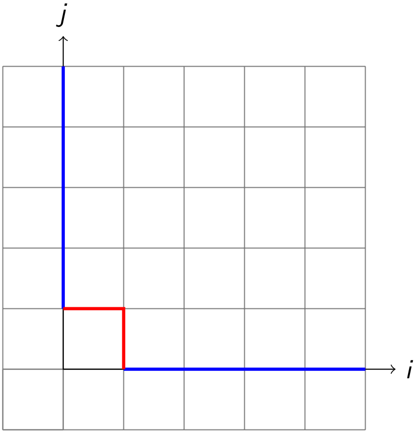

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By induction, $X(i,j), Y(i,j)$ are mut. indep. along down-right paths

$$X(i,j) \sim \text{Exp}(a_i + z)$$
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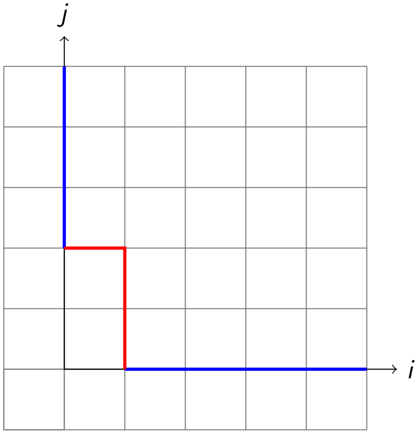
(Burke property)

Stationary model



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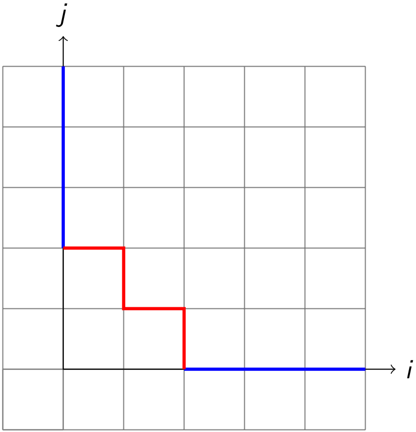

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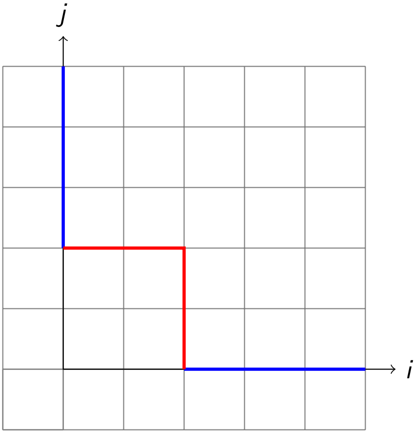

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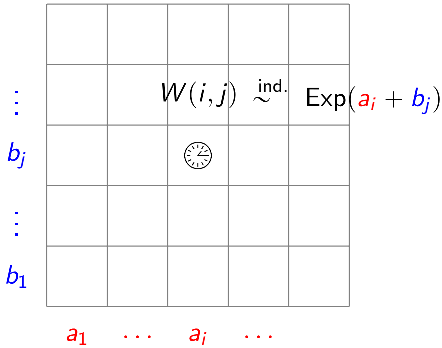
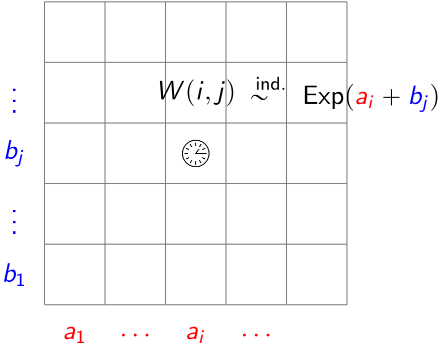
(Burke property)

Random environment



$\mathbf{a} = (a_i)$, $\mathbf{b} = (b_j)$ indep. i.i.d., finite mean $\geqslant c > 0$ sequences (can be weaker)

Random environment



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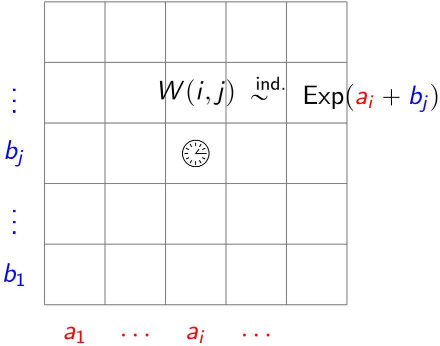
$\mathbf{P}_{\mathbf{a},\mathbf{b}}$: conditioned on $(\mathbf{a}, \mathbf{b})$,

$$W(i,j) \overset{ind.}{\sim} \text{Exp}(a_i + b_j).$$



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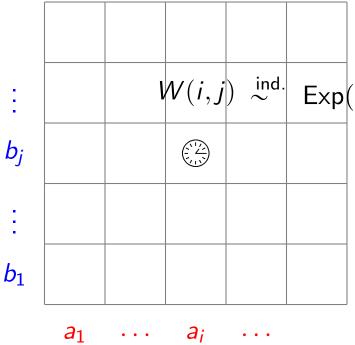
$\mathbf{P_{a,b}}$: conditioned on $(\mathbf{a}, \mathbf{b})$,

$$W(i,j) \overset{ind.}{\sim} \text{Exp}(a_i + b_j).$$

$\mathbb{P}$: average $\mathbf{P_{a,b}}$ over $(\mathbf{a}, \mathbf{b})$.

Random environment



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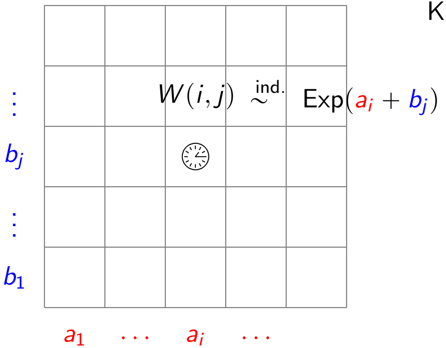
Key points:

Random environment



Key points:

$\mathbf{P_{a,b}}$: indep., not ident. dist.:
if $i \neq i'$ or $j \neq j'$

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$$W(i,j) \perp W(i',j')$$

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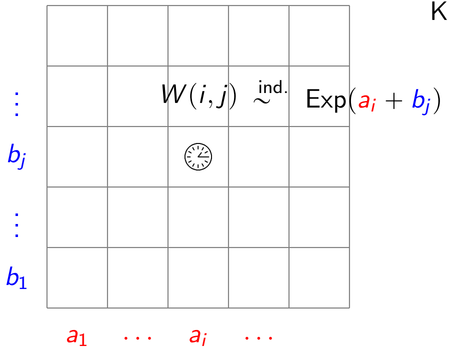

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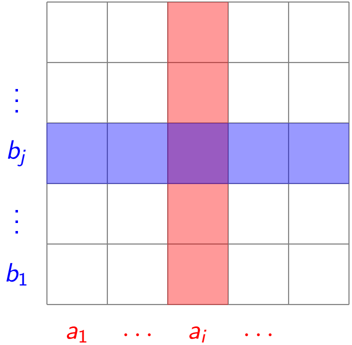

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Shape function

Theorem (Emrah '15)

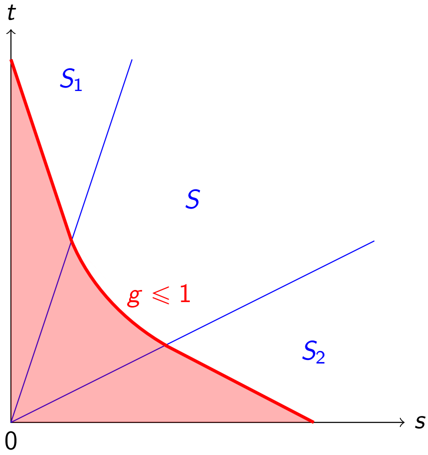
For $s, t > 0$, $\mathbb{P}$ almost surely and for almost all $(\mathbf{a}, \mathbf{b})$ $\mathbf{P}_{\mathbf{a},\mathbf{b}}$ almost surely

$$\lim_{n\to\infty} n^{-1} G(\lfloor ns \rfloor, \lfloor nt \rfloor) = \min_{-\underline{\alpha}\leqslant z\leqslant\underline{\beta}} \left\{ s\,\mathrm{E}\left[\frac{1}{a_1+z}\right] + t\,\mathrm{E}\left[\frac{1}{b_1-z}\right]\right\}$$

Key points:

- Only depends on marginal distributions of a_1 and b_1 separately. Notation: $\underline{\alpha} = \operatorname{essinf}\{a_1\}$, $\underline{\beta} = \operatorname{essinf}\{b_1\}$.
- Tractable 1D minmization problem.
- $g_z(s,t) = s\,\mathrm{E}\left[(a_1+z)^{-1}\right] + t\,\mathrm{E}\left[(b_1-z)^{-1}\right]$ is the shape function in the stationary version of the model.

Asymptotic shape of the cluster

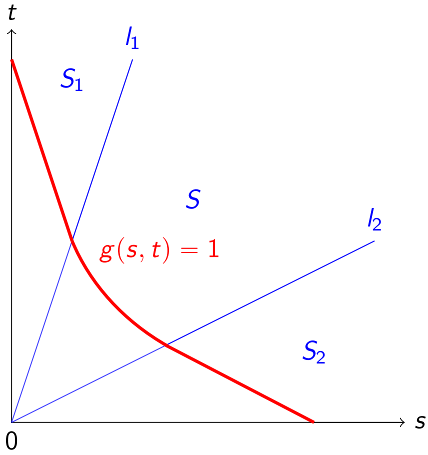


- g is strictly concave in S, linear in S_1 and S_2.
- $S_1, S_2 \neq \varnothing$ iff

$$E\big[(a_1 - \underline{\alpha})^{-2}\big] < \infty \quad (S_1)$$

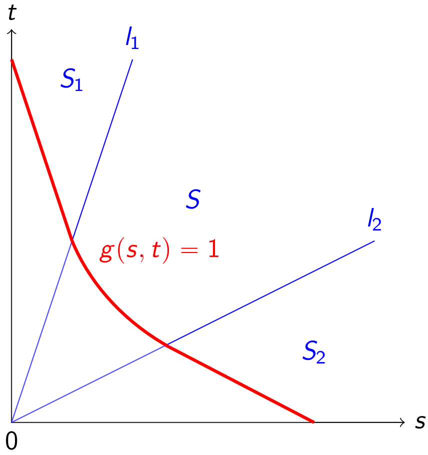
$$E\big[(b_1 - \underline{\beta})^{-2}\big] < \infty \quad (S_2)$$

Right tail large deviations



Pick a direction (s, t). Interested in

Right tail large deviations



Pick a direction (s, t). Interested in

$$r > g(s,t)$$

$$\mathbf{P}(n^{-1}G(\lfloor ns \rfloor, \lfloor nt \rfloor) \geqslant r) \approx e^{-n\,\mathbf{J}_{s,t}(r)}$$

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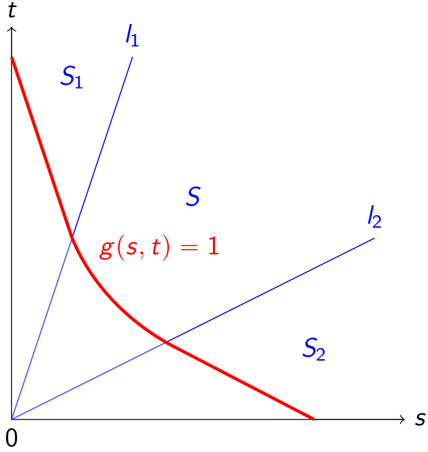
Right tail large deviations



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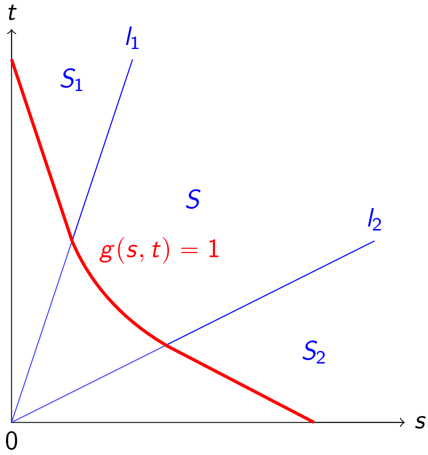
Is there a difference between $(s,t) \in S$ and $(s,t) \in S_1, S_2$? What happens when $(s,t) \in l_1, l_2$?

Right tail large deviations



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Is there a difference between $(s,t) \in S$ and $(s,t) \in S_1, S_2$? What happens when $(s,t) \in l_1, l_2$?

Is there a relationship between the $\mathbf{J}_{s,t}(r)$ and $\mathbb{J}_{s,t}(r)$?

Theorem

For almost all $(\mathbf{a}, \mathbf{b})$, *for any* $s, t > 0$ *and* $r \geqslant g(s,t)$

$$\mathbf{J}_{s,t}(r) = \lim_{n\to\infty} -n^{-1} \log \mathbf{P}_{\mathbf{a},\mathbf{b}} \left(n^{-1} G(\lfloor ns \rfloor, \lfloor nt \rfloor) \geqslant r \right)$$

$$= \sup_{\substack{\lambda\in(0,\underline{\alpha}+\underline{\beta}) \\ z\in(-\underline{\alpha},\underline{\beta}-\lambda)}} \left\{ r\lambda - s\,\mathrm{E}\log\left(\frac{a_1+z+\lambda}{a_1+z}\right) - t\,\mathrm{E}\log\left(\frac{b_1-z}{b_1-z-\lambda}\right) \right\}$$

Key points:

- Only depends on marginal distributions of a_1 and b_1 separately.
- Tractable 2D maximization problem.
- LDP with rate function $\mathbf{I}_{s,t}(r) = \mathbf{J}_{s,t}(r)1_{\{r\geqslant g(s,t)\}} + \infty 1_{\{r<g(s,t)\}}$. (Left tail rate should be n^2 under mild conditions).

Example rate functions

If $a_1, b_1 \sim \delta_{\frac{c}{2}}$ then for $r \geqslant g(s,t) = c^{-1}(\sqrt{s} + \sqrt{t})^2$ we have

$$\mathbf{J}_{s,t}(r) =$$
$$\sqrt{(s+t-cr)^2 - 4st} - 2s\cosh^{-1}\left(\frac{s-t+cr}{2\sqrt{csr}}\right) - 2t\cosh^{-1}\left(\frac{t-s+cr}{2\sqrt{ctr}}\right)$$

which recovers a result of Seppäläinen ’98.

Example rate functions

If $a_1, b_1 \sim \mathrm{Unif}[c/2, c/2+l]$, then for $r \geqslant g(s,s) = \frac{2s}{l} \log\left(1 + \frac{2l}{c}\right)$

$$\mathbf{J}_{s,s}(r) = r\,\lambda_\star - \frac{2s}{l}\int_{c/2}^{c/2+l} \log\left(\frac{x + z_\star + \lambda_\star}{x + z_\star}\right) dx$$

where

$$z_\star = -\sqrt{\frac{(c/2+l)^2 - c^2 e^{rl/s}/4}{1 - e^{rl/s}}}$$

$$z_\star + \lambda_\star = \sqrt{\frac{(c/2+l)^2 - c^2 e^{rl/s}/4}{1 - e^{rl/s}}}.$$

Example rate functions

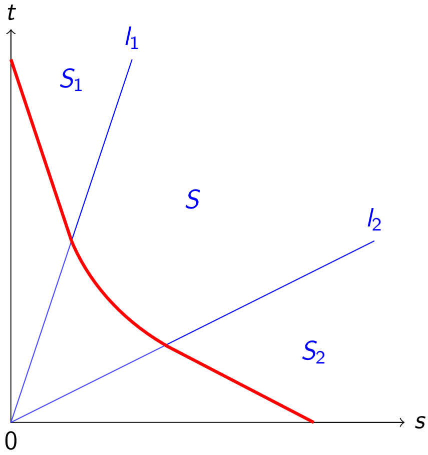
If $a_1, b_1 \sim p\delta_c + q\delta_d$, $0 \leqslant p \leqslant 1$, $q = 1 - p$, then for $r \geqslant g(s,s) = 2s\left(pc^{-1} + qd^{-1}\right)$,

$$\mathbf{J}_{s,s}(r) = r\,\lambda_\star - sp\log\left(\frac{c + z_\star + \lambda_\star}{c + z_\star}\right) - tq\log\left(\frac{c - z_\star}{c - z_\star - \lambda_\star}\right) - sq\log\left(\frac{d + z_\star + \lambda_\star}{d + z_\star}\right) - tq\log\left(\frac{d - z_\star}{d - z_\star - \lambda_\star}\right)$$

where

$$z_\star = \frac{2cp + 2dq + c^2r + d^2r - \sqrt{\Delta}}{2r}$$
$$z_\star + \lambda_\star = \frac{2cp + 2dq + c^2r + d^2r + \sqrt{\Delta}}{2r},$$
$$\Delta = (2cp + 2dq + c^2r + d^2r)^2 + 4r(2cd^2p + 2c^2dq - c^2d^2r).$$

Expected fluctuations



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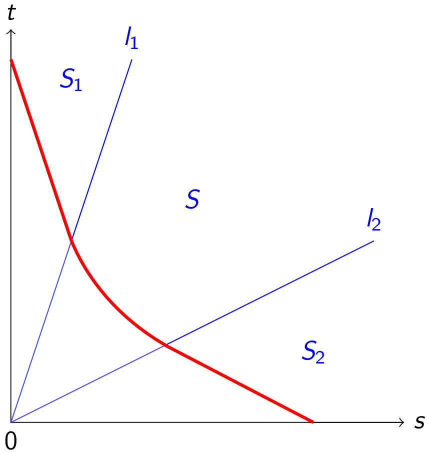

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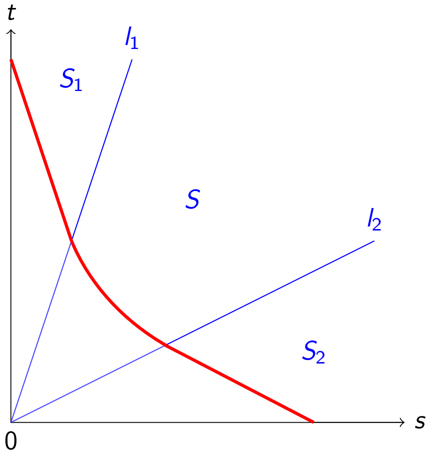

- Quenched fluct. should be TW_{GUE} in S, but not in S_1, S_2.
- Q1: Can we "see" different scaling exponents in the rate functions?
- Q2: What happens when $(s, t) \in l_1, l_2$?

Scaling and the quenched rate functions

Notation: $\zeta \in [-\underline{\alpha}, \underline{\beta}]$ solves (uniquely) $g_\zeta(s,t) = g(s,t)$

Proposition

For any $s, t > 0$, *as* $\epsilon \downarrow 0$, $\mathbf{J}_{s,t}(g(s,t) + \epsilon) =$

$$\begin{cases} \left(-s\,\mathrm{E}\left[\dfrac{2}{(a_1 - \underline{\alpha})^2}\right] + t\,\mathrm{E}\left[\dfrac{2}{(b_1 + \underline{\alpha})^2}\right]\right)^{-1} \epsilon^2 + o(\epsilon^2) & (s,t) \in S_1 \\ \dfrac{2}{3}\left(s\,\mathrm{E}\left[\dfrac{1}{(a_1 - \underline{\alpha})^3}\right] + t\,\mathrm{E}\left[\dfrac{1}{(b_1 + \underline{\alpha})^3}\right]\right)^{-1/2} \epsilon^{3/2} + o(\epsilon^{3/2}) & (s,t) \in I_1 \\ \dfrac{4}{3}\left(s\,\mathrm{E}\left[\dfrac{1}{(a_1 + \zeta)^3}\right] + t\,\mathrm{E}\left[\dfrac{1}{(b_1 - \zeta)^3}\right]\right)^{-1/2} \epsilon^{3/2} + o(\epsilon^{3/2}) & (s,t) \in S \\ \dfrac{2}{3}\left(s\,\mathrm{E}\left[\dfrac{1}{(a_1 + \underline{\beta})^3}\right] + t\,\mathrm{E}\left[\dfrac{1}{(b_1 - \underline{\beta})^3}\right]\right)^{-1/2} \epsilon^{3/2} + o(\epsilon^{3/2}) & (s,t) \in I_2 \\ \left(s\,\mathrm{E}\left[\dfrac{2}{(a_1 + \underline{\beta})^2}\right] - t\,\mathrm{E}\left[\dfrac{2}{(b_1 - \underline{\beta})^2}\right]\right)^{-1} \epsilon^2 + o(\epsilon^2) & (s,t) \in S_2 \end{cases}$$

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Take $(s,t) \in S$ and set

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For n large and large enough r (but not $\mathrm{O}(n^{\frac{2}{3}})$), we might expect

$$\mathbf{P}_{\mathbf{a},\mathbf{b}}(G(\lfloor ns \rfloor, \lfloor nt \rfloor) - ng(s,t) \geqslant n^{\frac{1}{3}} C^{\frac{1}{3}} r)$$

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$$\approx e^{-\frac{4}{3} C^{-\frac{1}{2}} (C^{\frac{1}{3}} n^{-\frac{2}{3}} r)^{\frac{3}{2}} n} = e^{-\frac{4}{3} r^{\frac{3}{2}}}$$

which agrees with the leading order TW_{GUE} right tail.

Annealed large deviations

Theorem

For $s, t > 0$ and $r \geqslant g(s,t)$,

$$\mathbb{J}_{s,t}(r) = \lim_{n\to\infty} -n^{-1}\log\mathbb{P}\left(n^{-1}G(\lfloor ns\rfloor,\lfloor nt\rfloor)\geqslant r\right)$$

$$= \sup_{\substack{\lambda\in(0,\underline{\alpha}+\underline{\beta})\\ z\in(-\underline{\alpha},\underline{\beta}-\lambda)}} \left\{ r\lambda - s\log \mathrm{E}\left[\frac{a_1+z+\lambda}{a_1+z}\right] - t\log\mathrm{E}\left[\frac{b_1-z}{b_1-z-\lambda}\right]\right\}$$

Key points

- Very similar to quenched rate function - only difference is $\mathrm{E} \leftrightarrow \log$
- Variational problem still tractable (though no explicit examples)
- For all $s, t > 0$, $\mathbb{J}_{s,t}(g(s,t)+\epsilon) = C_{s,t}\epsilon^2 + o(\epsilon^2)$ ($C_{s,t}$ explicit)
- There are rate n left tail $(n^{-1}G(\lfloor ns\rfloor,\lfloor nt\rfloor)) < g(s,t)-\epsilon)$ annealed large deviations.

Variational connection for right tail

Theorem

For any $s, t > 0$ and $r \geqslant g(s,t)$,

$$\mathbb{J}_{s,t}(r) = \inf_{\nu_1,\nu_2} \left\{ \mathbf{I}_{s,t}^{\nu_1,\nu_2}(r) + s\,\mathrm{H}(\nu_1|\alpha) + t\,\mathrm{H}(\nu_2|\beta) \right\}.$$

There exists a unique (explicit) minimizing pair (ν_1, ν_2) for which equality holds.

Key points:

- Notation: α is the dist. of a_1, β is the dist of b_1, $\mathrm{H}(\cdot|\cdot)$ is relative entropy.
- $\mathbf{I}_{s,t}^{\nu_1,\nu_2}(r)$ is quenched rate function for the model with marginals $a_1 \sim \nu_1$, $b_1 \sim \nu_2$.

Thanks!