Parabolic Anderson model & large time asymptotic

Jingyu Huang
(joint work with Khoa Lê and David Nualart)

Department of Mathematics
the University of Utah

May 9, 2016

Linear Stochastic Heat Equation

Consider stochastic heat equation (SHE)

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\dot{W}, \quad u(0,x) = u_0(x), \quad t \geq 0, x \in \mathbb{R}^\ell.$$

$\dot{W} = \dot{W}(t,x)$ is a generalized Gaussian field with covariance

$$\mathbb{E}[\dot{W}(s,x)\dot{W}(t,y)] = \delta(t-s)\gamma(x-y)$$

- Existence and uniqueness of a random field solution
- Large time asymptotic of the nth moments $\mathbb{E}u(t,x)^n$
- Exponential growth indices

Intermittency - Lyapunov exponent

Intermittency: most of the mass is concentrated on small islands.

Let $u_0 = 1$, we define the *Lyapunov exponent* of u (if it exists):

$$\mathcal{E}_n := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} u^n(t,x)$$

We say that the solution is *intermittent* if the map $n \mapsto \frac{\mathcal{E}_n}{n}$ is strictly increasing: as time time gets large, the moments of the solution grows very fast

Question: If we start with a compactly support function (say $1_{[-1,1]}$), are there regions (of x) where the moment of the solution does not grow exponentially fast?

- Exponential growth indices

Space-time white noise in 1D: $\mathbb{E}[\dot{W}(s,x)\dot{W}(t,y)] = \delta(s-t)\delta(x-y)$

1. Existence and uniqueness: solution has finite moments and a.s. continuous (J. Walsh'86, L. Chen-R. Dalang'14)
2. Intermittency: $u_0 = 1$, $\mathbb{E}|u^n(t,x)| \approx C_1 e^{C_2 n^3 t}$
3. Exact Lyapunov exponent: $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}u^n(t,x) = \frac{n(n^2-1)}{24}$, $u_0 = 1$ (X. Chen'15)
4. D. Conus-D. Khoshnevisan'10: u_0 has compact support, there exist $\frac{1}{2\pi} \le \underline{\alpha}^c \le \overline{\alpha}^c \le \frac{1}{2}$ such that

$$\texttt{If} \quad \alpha > \overline{\alpha}^c : \quad \limsup_{t\to\infty} \frac{1}{t} \log \sup_{|x|\ge \alpha t} \mathbb{E}u^2(t,x) < 0$$

$$\texttt{If} \quad \alpha < \underline{\alpha}^c : \quad \liminf_{t\to\infty} \frac{1}{t} \log \sup_{|x|\ge \alpha t} \mathbb{E}u^2(t,x) > 0$$

L. Chen-R. Dalang'14: $\underline{\alpha}^c = \overline{\alpha}^c = \frac{1}{2}$ (exponential growth index)

Our assumptions

- Noise: $\mathbb{E}[\dot{W}(s,x)\dot{W}(t,y)] = \delta(s-t)\gamma(x-y)$
- The covariance γ is nonnegative and nonnegative definite.
- The Fourier transform of γ is a tempered measure $\mu(d\xi)$, such that

$$\int_{\mathbb{R}^\ell} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty \qquad \text{(Dalang)}$$

- The *initial data* satisfies

$$p_t * |u_0|(x) < \infty$$

for all $t > 0$ and $x \in \mathbb{R}^\ell$.

Examples of covariance

1. Riesz-type:

$$\gamma(x) = |x|^{-\eta}, \quad 0 < \eta < 2 \wedge \ell$$

2. Fractional Brownian motion (regular):

$$\gamma(x) = \prod_{j=1}^{\ell} |x_j|^{2H_j - 2}, \quad \frac{1}{2} < H_j < 1, \quad \sum_{j=1}^{\ell} H_j > \ell - 1$$

3. Brownian motion (white noise): $\ell = 1$

$$\gamma(x) = \delta(x), \quad \mu(d\xi) = d\xi$$

Stochastic integration with respect to W

$\mathcal{H}$: the Hilbert space with inner product

$$\langle g, h\rangle_{\mathcal{H}} = \frac{1}{(2\pi)^\ell}\int_{\mathbb{R}^\ell} \hat{g}(\xi)\overline{\hat{h}(\xi)}\mu(d\xi).$$

W: *isonormal Gaussian* process $\{W(\phi), \phi \in L^2(\mathbb{R}_+, \mathcal{H})\}$ parametrized by the Hilbert space $L^2(\mathbb{R}_+, \mathcal{H})$.
$\mathcal{F}_t$: σ-algebra generated by W up to time t.
Elementary process g:

$$g(s) = \sum_{i=1}^{n}\sum_{j=1}^{m} X_{i,j}\, \mathbf{1}_{(a_i,b_i]}(s)\, \phi_j,$$

where $\phi_j \in \mathcal{H}$ and $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables
The integral of such a process with respect to W is defined as

$$\int_0^\infty \int_{\mathbb{R}^\ell} g(s,x)\, W(ds,dx) = \sum_{i=1}^{n}\sum_{j=1}^{m} X_{i,j}\, W\left(\mathbf{1}_{(a_i,b_i]} \otimes \phi_j\right).$$

Itô isometry:

$$\mathbb{E}\left|\int_0^\infty \int_{\mathbb{R}^\ell} g(s,x)\, W(ds,dx)\right|^2 = \mathbb{E}\int_0^\infty \|g(s,\cdot)\|_{\mathcal{H}}^2 ds$$
$$= \frac{1}{(2\pi)^\ell}\mathbb{E}\int_0^\infty \int_{\mathbb{R}^\ell} |\hat{g}(s,\xi)|^2 \mu(d\xi) ds$$

We can extend the stochastic integration from elementary processes to all adapted processes as long as the right hand side is finite.

Mild solution - Chaos expansion

- Mild solution (mimic Duhamel principle)

$$u(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}^\ell} p_{t-s}(x-y)u(s,y)W(ds,dy)$$

- Iterate this equation, the solution (if exists) has the form (chaos expansion)

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot;t,x))$$

  - I_n is the n-th multiple integral w.r.t W
  - f_n's are deterministic kernels which can be explicitly written.
  - $\mathbb{E}\left(I_n(f_n(\cdot;t,x))^2\right) = n!\|f_n(\cdot;t,x)\|^2_{(L^2(\mathbb{R}_+;\mathcal{H}))^{\otimes n}}$

Existence and uniqueness - Moment representation

- *Existence and uniqueness*

$$\iff \mathbb{E}u^2(t,x) = \sum_{n=1}^{\infty} n! \|f_n(\cdot; t, x)\|^2_{(L^2(\mathbb{R}_+;\mathcal{H}))^{\otimes n}} < \infty\,.$$

- *Moment representation:* Feynman-Kac formula for n-th moment

$$\mathbb{E}\left[\prod_{j=1}^{n} u(t,x_j)\right] = \mathbb{E}\left[\prod_{j=1}^{n} u_0(B^j(t)+x_j) \times \exp\left\{\sum_{1\le j<k\le n}\int_0^t \gamma(B^j(s)-B^k(s)+x_j-x_k)ds\right\}\right]$$

where B^j's are independent Brownian motions in $\mathbb{R}^\ell$.

Large time asymptotic, $u_0 = 1$

If $u_0 = 1$, the moment representation is

$$\mathbb{E}u^n(t,x) = \mathbb{E}\exp\left\{\int_0^t \sum_{1\le j<k\le n} \gamma\left(B^j(s) - B^k(s)\right) ds\right\}$$

It is well-known (X. Chen) that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}u^n(t,x) = \mathcal{E}_n(\gamma)$$

where

$$\mathcal{E}_n(\gamma) = \sup_{g:\|g\|_{L^2(\mathbb{R}^{n\ell})}=1} \left\{ \sum_{1\le j<k\le n} \int_{\mathbb{R}^{n\ell}} \gamma(x_j - x_k) g^2(x) dx - \frac{1}{2}\int_{\mathbb{R}^{n\ell}} |\nabla g(x)|^2 dx \right\}$$

Moment representation with respect to Brownian bridges

Using the decomposition $B(s) = \underbrace{B(s) - \frac{s}{t}B(t)}_{\text{Brownian bridge } B_{0,t}(s)} + \frac{s}{t}B(t)$, we obtain

$$\mathbb{E}\left[\prod_{j=1}^{n} u(t, x_j)\right] = \int_{(\mathbb{R}^{\ell})^n} F(t, x, y) \prod_{j=1}^{n} [u_0(x_j + y_j) p_t(y_j)] dy_1 \cdots dy_n .$$

where $F(t, x, y)$ is the function defined by

$$\mathbb{E} \exp\left\{ \int_0^t \sum_{1 \le j < k \le n} \gamma\left(B_{0,t}^j(s) - B_{0,t}^k(s) + x_j - x_k + \frac{s}{t}(y_j - y_k) \right) ds \right\}$$

$B_{0,t}^1, \ldots, B_{0,t}^n$ are independent Brownian bridges over the time interval $[0, t]$ which start and end at 0.

Large time asymptotic

H-Lê-Nualart'16

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a bounded continuous function. Then for every fixed $x_0 \in \mathbb{R}^n$

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{|y|\le o(1)t} \mathbb{E} \exp\left\{ \int_0^t F\left(B_{0,t}(s) + x_0 + \frac{s}{t} y\right) ds \right\}$$
$$= \lim_{t\to\infty} \frac{1}{t} \log \inf_{|y|\le o(1)t} \mathbb{E} \exp\left\{ \int_0^t F\left(B_{0,t}(s) + x_0 + \frac{s}{t} y\right) ds \right\}$$
$$= \mathcal{E}(F),$$

where

$$\mathcal{E}(F) = \sup_{g:\|g\|_{L^2}=1} \left\{ \int_{\mathbb{R}^n} F(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^n} |\nabla g(x)|^2 dx \right\}.$$

Large time asymptotic, general initial condition

H-Lê-Nualart'16

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{(x_1,\dots,x_n)\in(\mathbb{R}^\ell)^n} \frac{\mathbb{E}\left[\prod_{j=1}^n u(t,x_j)\right]}{\prod_{j=1}^n (p_t * u_0)(x_j)} \le \mathcal{E}_n(\gamma)\,,$$

and for every $M, M' > 0$

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{(x_1,\dots,x_n)\in A_{M'}} \frac{\mathbb{E}\left[\prod_{j=1}^n u(t,x_j)\right]}{\int_{A_M} \prod_{j=1}^n p_t(y_j) u_0(x_j+y_j) dy} \ge \mathcal{E}_n(\gamma)\,,$$

where

$$A_M = \{(y_1,\dots,y_n) \in (\mathbb{R}^\ell)^n : |y_j - y_k| \le M \text{ for all } 1 \le j < k \le n\}$$

Proof of upper bound, assuming $n = 2$ for simplicity

Start from moment representation

$$\mathbb{E}[u(t,x_1)u(t,x_2)] = \int_{(\mathbb{R}^\ell)^2} F(t,x,y) \prod_{j=1}^{2} [u_0(x_j + y_j) p_t(y_j)] dy_1 dy_2 \,.$$

where

$$F(t,x,y) = \mathbb{E} \exp \left\{ \int_0^t \gamma\Big(B^1_{0,t}(s) - B^2_{0,t}(s) + x_1 - x_2 + \frac{s}{t}(y_1 - y_2) \Big) ds \right\}$$

Since γ is positive definite, we have

$$F(t,x,y) \leq F(t,0,0) \Rightarrow \frac{\mathbb{E}[u(t,x_1)u(t,x_2)]}{p_t * u_0(x_1) p_t * u_0(x_2)} \leq F(t,0,0)$$

Then we apply large deviation result for exponential functional of Brownian bridges.

The lower bound is obtained by similar idea although it is more technical. □

Exponential growth indices - u_0 has compact support

H-Lê-Nualart'16

For every $\alpha > 0$, we have

$$\Gamma(\alpha) := \lim_{t\to\infty} \frac{1}{t} \log \sup_{|x|\geq\alpha t} \mathbb{E}u^n(t,x) = \mathcal{E}_n(\gamma) - \frac{n\alpha^2}{2}.$$

In particular,

$$\Gamma(\alpha) < 0 \quad \texttt{if} \quad \alpha > \alpha_n^c$$

$$\Gamma(\alpha) > 0 \quad \texttt{if} \quad \alpha < \alpha_n^c$$

where

$$\alpha_n^c = \sqrt{\frac{2\mathcal{E}_n(\gamma)}{n}}$$

Colored in time

What happens if the noise has some time covariance?

$$\mathbb{E}\left[\dot{W}(s,x)\dot{W}(t,y)\right] = \gamma_0(s-t)\gamma(x-y)\,.$$

- γ_0 and γ are nonnegative and nonnegative definite, locally integrable functions.
- γ has Fourier transform $\mu(d\xi)$.
- $p_t * |u_0|(x) < \infty$ and Dalang's condition $\iff$ existence and uniqueness of the solution.
- Moment formula: (Hu-H-Nualart-Tindel' 15)

$$\begin{aligned}\mathbb{E}\left[u(t,x)^n\right] &= \mathbb{E}_B\Bigg[\prod_{i=1}^n u_0(B_t^i + x)\\ &\quad \exp\Bigg(\sum_{1\le i<j\le n}\int_0^t\int_0^t \gamma_0(s-r)\gamma(B_s^i - B_r^j)dsdr\Bigg)\Bigg]\,.\end{aligned}$$

Constant initial condition

Chen' 2015

Assume that the initial condition is $u_0 = 1$. Choose $\gamma_0(t) = |t|^{-\alpha_0}$ and $\gamma(x) = |x|^{-\alpha}$ with $0 < \alpha_0 < 1$ and $0 < \alpha < 2 \wedge d$. Then

$$\lim_{t\to\infty} t^{-a} \log \mathbb{E} u(t,x)^n = n \left(\frac{n-1}{2}\right)^{\frac{2}{2-\alpha}} \mathcal{E}$$

for every $x \in \mathbb{R}^\ell$. Where $a = \frac{4-\alpha-2\alpha_0}{2-\alpha}$ and $\mathcal{E}$ is some constant.

Intermittency front

Define

$$\lambda_*(n) = \sup\left\{\lambda > 0 : \liminf_{t\to\infty} t^{-a} \sup_{|x|\geq \lambda t^{\frac{a+1}{2}}} \log \mathbb{E}|u(t,x)|^n > 0\right\}$$

and

$$\lambda^*(n) = \inf\left\{\lambda > 0 : \limsup_{t\to\infty} t^{-a} \sup_{|x|\geq \lambda t^{\frac{a+1}{2}}} \log \mathbb{E}|u(t,x)|^n < 0\right\}.$$

H-Lê-Nualart'16

Assume that u_0 is compactly supported and uniformly bounded below in a ball of radius M, then

$$a^{\frac{a}{2}}(a+1)^{-\frac{a+1}{2}}\left(2\left(\frac{n-1}{2}\right)^{\frac{2}{2-\alpha}}\mathcal{E}\right)^{1/2} \leq \lambda_*(n)$$

$$\leq \lambda^*(n) \leq \left(2\left(\frac{n-1}{2}\right)^{\frac{2}{2-\alpha}}\mathcal{E}\right)^{1/2}.$$

- Conjecture: Upper bound is optimal.

Thank you.