Tracy-Widom fluctuations for the corner growth model with inhomogeneous geometric weights

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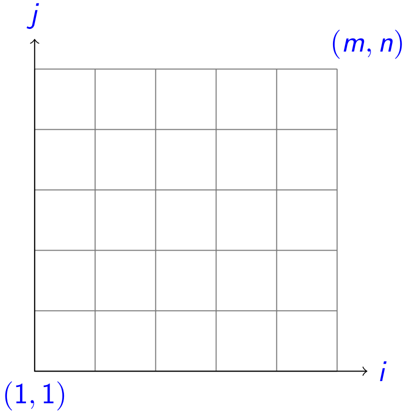
Corner growth model



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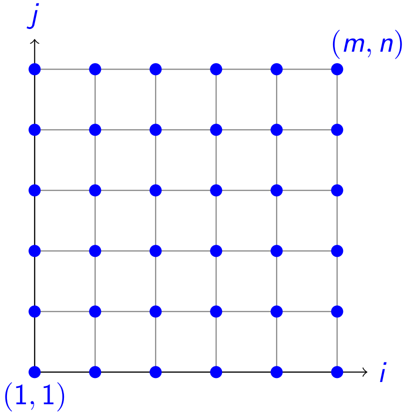

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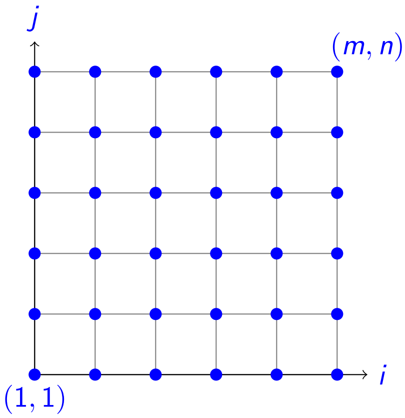

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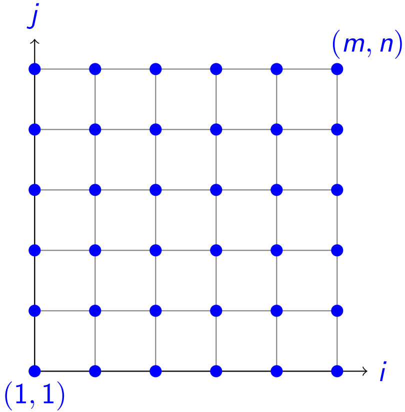

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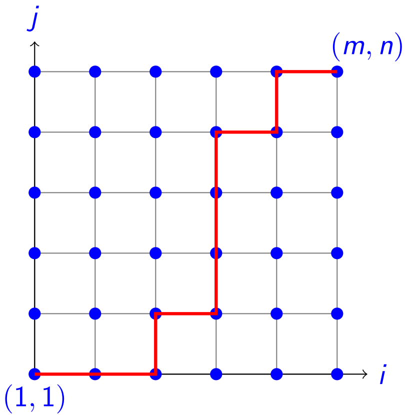

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- The conjecture should hold for a *large* class of $\mathbf{P}$. It has been verified when the weights are i.i.d. exponential or geometric. **[Johansson '00]**

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- Law of large numbers. **[Cohn, Elkies, Propp '96], [Josckusch, Propp, Shor '98], [Seppäläinen '98]**

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for $r > 0$.

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where

$$\sigma(r) = \frac{1}{1-q}\left(\frac{q}{r}\right)^{1/6}(\sqrt{q} + \sqrt{r})^{2/3}(1 + \sqrt{qr})^{2/3}.$$

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where $\sigma(r) = r^{-1/6}(1+r)^{4/3}$.

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$$\mathbf{P} \overset{RSK}{\Rightarrow} \boxed{\begin{array}{l}\text{Schur measures } \textbf{[Okounkov '01]} \\ \text{Determinantal point processes}\end{array}} \Rightarrow \boxed{\begin{array}{l}\text{Fredholm determinants} \\ \text{for } \mathbf{P}(G(m,n) \leqslant k)\end{array}}$$

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- Our results hold for a.e realization of $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$.

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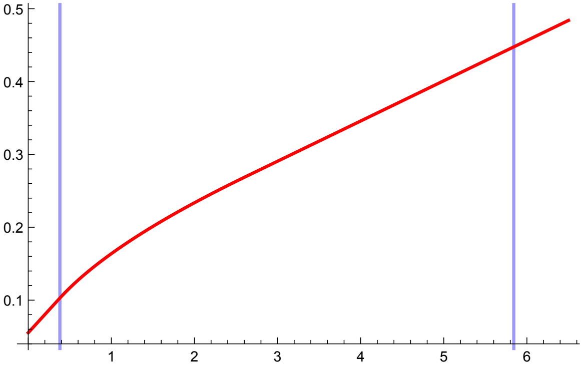

Figure: Plot of $r \mapsto \gamma(r)$ (red) when $\alpha = \frac{1}{9}(1-a)^8 da$ and $\beta = \frac{1}{4068}(\frac{1}{2} - b)^8 \mathbf{1}_{b \leqslant \frac{1}{2}}\, db$. $c_1 \approx 0.381$ and $c_2 \approx 5.842$. (blue)

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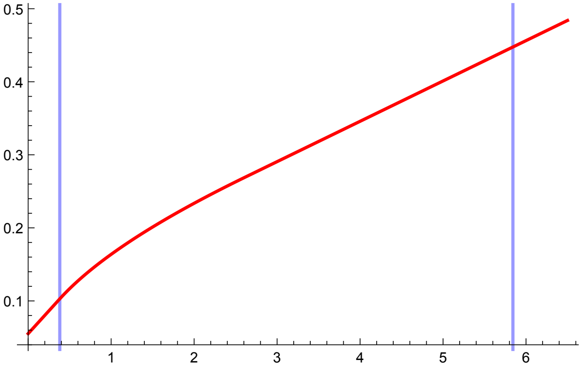

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- $c_1 = 0$ iff $\int \frac{\alpha(da)}{(\bar{\alpha}-a)^2} = \infty$ and $c_2 = \infty$ iff $\int \frac{\beta(db)}{(\bar{\beta}-b)^2} = \infty$.

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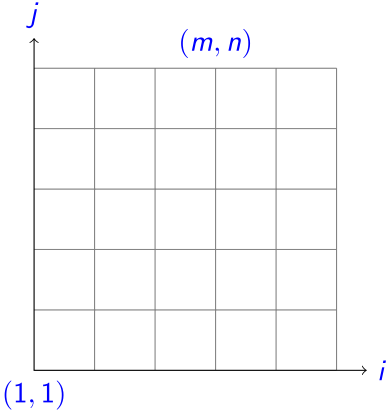

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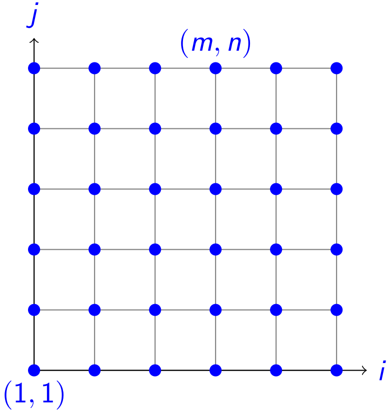

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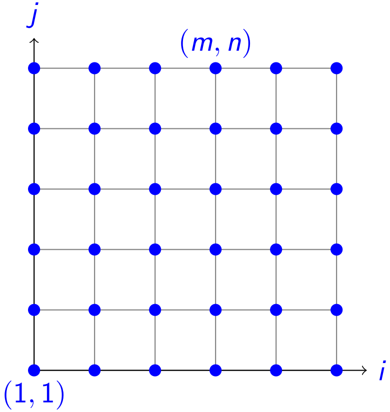

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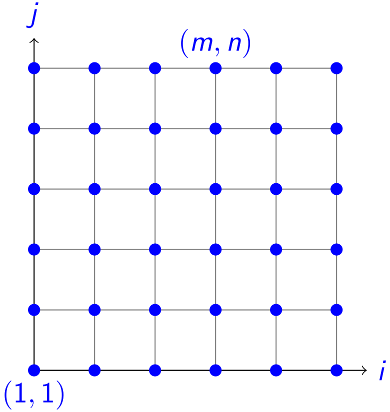

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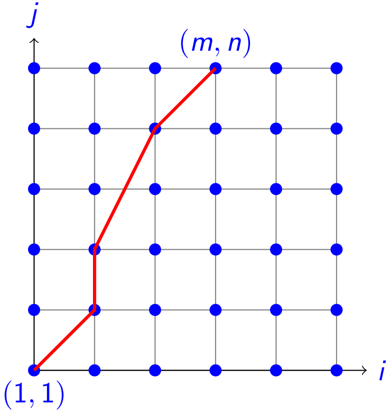

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- We presently observe Tracy-Widom fluctuations in the strictly concave region for the inhomogeneous CGM.

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  - By ergodicity, $\lim_{n\to\infty} \gamma_n = \gamma(r)$, $\lim_{n\to\infty} \zeta_n = \zeta(r)$ and $\lim_{n\to\infty} \sigma_n = \sigma(r)$.

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- If used $\gamma(r)$ instead of γ_n the limit equals 0 or 1 a.s.

One-point distribution for the last-passage times

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- For $m, n \in \mathbb{N}$, $x, y \in \mathbb{Z}_+$ and $z \in \mathbb{C} \setminus \{a_1, \ldots, a_m\}$, define

$$F_{m,n,x}^{(a_i),(b_j)}(z) = \frac{\prod_{j=1}^{n}(1 - zb_j)}{\prod_{i=1}^{m}(z - a_i)} \cdot z^{m+x},$$

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and the *correlation kernel*

$$K_{m,n}(x, y) = \frac{1}{(2\pi \mathbf{i})^2} \oint_{|w|=\rho} \oint_{|z|=\rho} \frac{F_{m,n,x}^{(a_i),(b_j)}(z) F_{n,m,y}^{(b_j),(a_i)}(w)}{1 - zw} \, dz \, dw,$$

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- For $m, n \in \mathbb{N}$ and $k \in \mathbb{Z}_+$,

$$\mathbf{P}(G(m, n) \leqslant k) = 1 + \sum_{l=1}^{\infty} \frac{(-1)^l}{l!} \sum_{\substack{x_1, \dots, x_l \in \mathbb{Z}_+ \\ x_i \geqslant k}} \det_{i,j \in [l]} [K_{m,n}(x_i, x_j)]$$

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Theorem

Let $s_0 \geqslant 0$. For a.e. $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$,

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$$= \det_{i,j\in[l]} [\mathrm{A}(s_i, s_j)]$$

uniformly in $s_1, \dots, s_l \in [-s_0, s_0]$ for any $l \in \mathbb{N}$.

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Theorem

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- $s_1, \ldots, s_l$ are only bounded from below.
- The LHS is nonnegative. $\det_{i,j\in[l]} [K_{m,n}(x_i, x_j)] \in [0, 1]$.

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Thanks very much!

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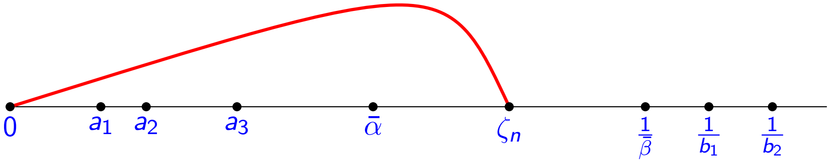
- Do steepest-descent analysis of $I_{\lfloor nr \rfloor, n, \lfloor n\gamma_n + n^{1/3}\sigma_n s \rfloor + l}$.
- Use averaging property (ergodicity) of $(a_i)_{i\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ to have uniform control.
- The same strategy used in **[Gravner, Tracy, Widom '02]** for the inhomogeneous Johansson-Seppäläinen model.

Steepest-descent analysis

Steepest-descent analysis

- $\log F_{m,n,x}(z) = -\sum_{i=1}^{m} \log(z - a_i) + \sum_{j=1}^{n} \log(1 - b_j z) + (x+m)\log z$

Steepest-descent analysis



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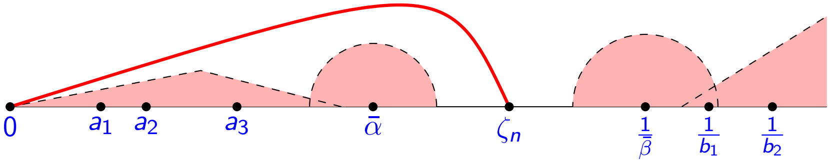
- A steepest-descent curve Φ_n emanates from ζ_n and ends at 0.

Steepest-descent analysis



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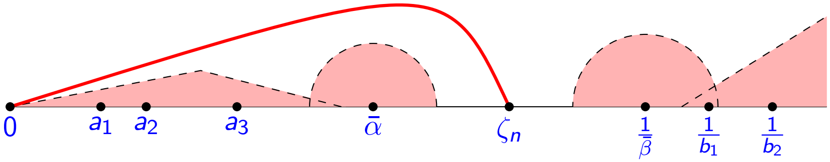
- Φ_n stays outside a fixed region containing the singularities.

Steepest-descent analysis



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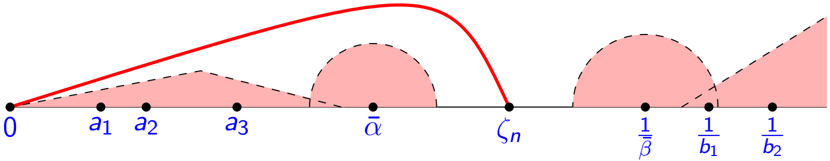
  - Enables uniform control of Φ_n in n.

Steepest-descent analysis



- $\log F_{m,n,x}(z) = -\sum_{i=1}^{m} \log(z-a_i) + \sum_{j=1}^{n} \log(1-b_j z) + (x+m)\log z$
- A steepest-descent curve Φ_n emanates from ζ_n and ends at 0.
- Φ_n stays outside a fixed region containing the singularities.
  - Enables uniform control of Φ_n in n.
- Use $\Phi_n + \bar{\Phi}_n$ as the contour of integration.

Thanks very much!