# Consistency of Modularity Clustering on Random Geometric Graphs

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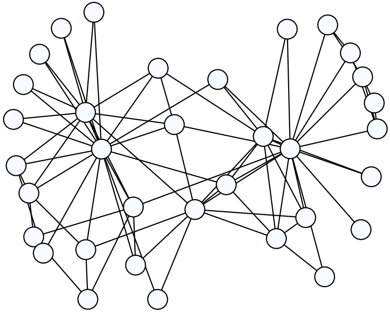
# Outline

# Graph Clustering

$$G = (X, W)$$



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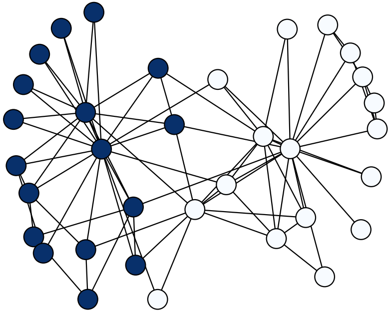

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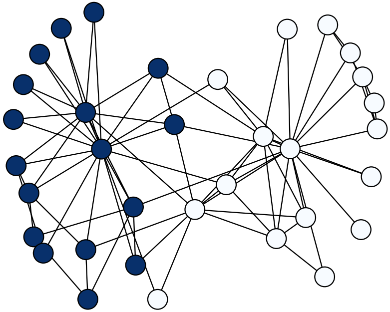
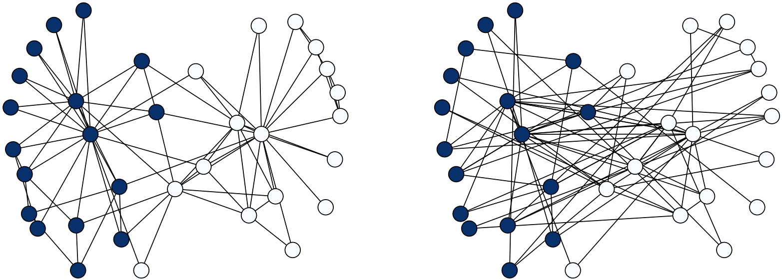


$$\frac{1}{2m} \sum_{i,j} W_{ij} \delta(c_i, c_j) \text{ where } m = \frac{1}{2} \sum_{i,j} W_{ij}.$$

# Modularity (Newman & Girvan '04)

$$Q(\mathcal{U}) = \frac{1}{2m}\sum_{i,j} W_{ij}\delta(c_i, c_j) - \frac{1}{2m}\sum_{i,j}\frac{d_i d_j}{2m}\delta(c_i, c_j)$$

where $d_i = \sum_j W_{ij}$, $\mathcal{U}$ a clustering.



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More generally ($\alpha$-modularity):

$$Q(\mathcal{U}) = \frac{1}{2m} \sum_{i,j} W_{ij}\delta(c_i, c_j) - \frac{1}{S^2} \sum_{i,j} d_i^\alpha d_j^\alpha \delta(c_i, c_j)$$

where $S = \sum_i d_i^\alpha$.

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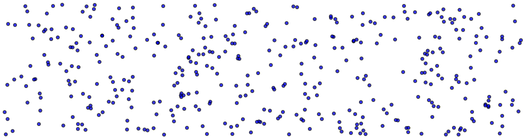

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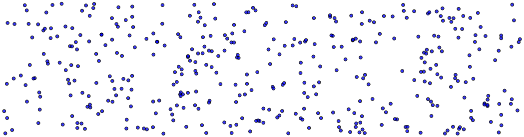

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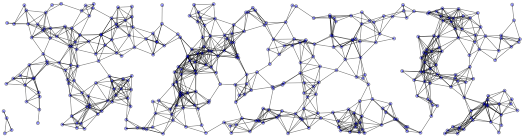

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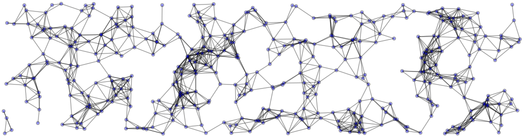

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Graph $\mathcal{G}_n = (\mathcal{X}_n, W)$.

# Questions

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Consistency: Subject to certain technical assumptions, $\mathcal{U}_n^* \to \mathcal{U}^*$ where $\mathcal{U}^*$ is a partition of $D$ characterized as the solution to a (deterministic) continuum optimization problem.

# Consistency of Clustering Methods

- K-Means (Pollard 1981)
- Spectral Clustering (von Luxburg, Belkin, & Bosquet 2008)
- Modularity (Bickel & Chen 2009, Zhao, Levina, & Zhu 2012)
- Cheeger Cut (García-Trillos, Slepčev, von Brecht, Laurent, & Bresson 2014)

# Outline

Introduction to Modularity Clustering

Pointwise Convergence

Convergence of Optimal Partitions

# Key Identity

Let $\mathcal{U}_n = \{U_{n,k}\}_{k=1}^{K}$ be a partition of $\mathcal{X}_n$, and let $u_{n,k} = \mathbb{1}_{U_{n,k}}$.

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$$1 - 1/K - Q_n(\mathcal{U}_n) = \frac{1}{S^2}\sum_{k=1}^{K}\Big(\sum_i d_i^{\alpha}\big[u_{n,k}(X_i) - 1/K\big]\Big)^2 + \epsilon_n \frac{n^2}{4m}\sum_{k=1}^{K} GTV_n(u_{n,k}),$$

where

$$GTV_n(u) := \frac{1}{\epsilon_n}\frac{1}{n^2}\sum_{i,j}\eta_{\epsilon_n}(X_i - X_j)|u(X_i) - u(X_j)|.$$

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Domain $D \subset \mathbb{R}^d$, fixed $K \geq 1$, and partition $\mathcal{U} = \{U_k\}_{k=1}^K$ of $D$.

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- The *perimeter* of $U_k$ in $D$, with respect to a weight $\rho^2$, is

$$\mathrm{Per}(U_k; \rho^2) = \int_{\partial U_k} \rho^2(x)\, d\mathcal{H}^{d-1}(x).$$

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More generally,

$$\mathrm{Per}(U_k; \rho^2) = TV(\mathbb{1}_{U_k}; \rho^2) := \sup_{\substack{\Phi \in C_c^1(D;\mathbb{R}^d) \\ |\Phi(x)| \leq \rho^2(x)}} \int_D \mathbb{1}_{U_k}(x) \mathrm{div}\, \Phi(x)\, dx.$$

*Remark:* When $f$ smooth, $TV(f; \rho^2) = \int_D |\nabla f| \rho^2(x)\, dx$.

# Continuum Partitioning

$$\mathcal{U}^* = \operatorname*{arg\,min}_{\substack{|\mathcal{U}|=K \\ \mu(U_k)=1/K}} \sum_{k=1}^{K} \mathrm{Per}(U_k; \rho^2).$$

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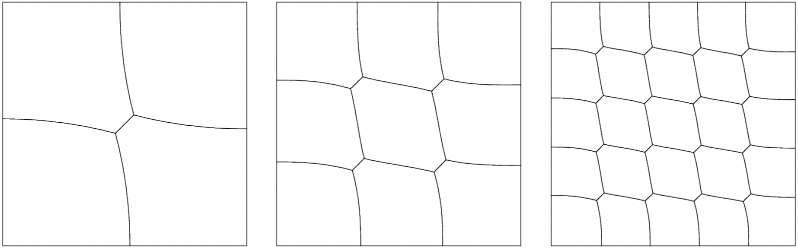

(a) $K = 4$ (b) $K = 9$. (c) $K = 25$.

Figure: Local minimizers on $D = (0,1)^2$, with $\rho(x) = 1$, $d\mu = dx$, produced using *The Surface Evolver*.

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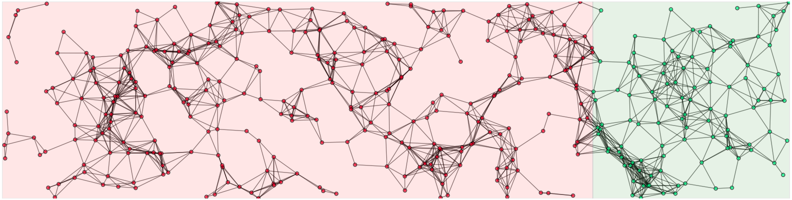

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## Theorem (Asymptotics)

*Let $\mathcal{U}$ be a finite perimeter partition. Suppose $\{\epsilon_n\}$ satisfies $\sum_{n=1}^{\infty} \exp(-n\epsilon_n^{(d+1)/2}) < \infty$ when $\alpha = 0, 1$ and $\sum_{n=1}^{\infty} \exp(-n\epsilon_n^{d}) < \infty$ otherwise.*

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$$\frac{1 - 1/K - Q_n(\mathcal{U}_n)}{\epsilon_n} \xrightarrow{a.s.} \begin{cases} C_{\eta,\rho} \sum_{k=1}^{K} \mathrm{Per}(U_k; \rho^2) & \text{if } \sum_{k=1}^{K} \Big(\mu(U_k) - 1/K\Big)^2 = 0, \\ \infty & \text{otherwise,} \end{cases}$$

*where*

$$d\mu(x) = \frac{\rho^{1+\alpha}(x)\,dx}{\int_D \rho^{1+\alpha}(x)\,dx} \quad \text{and} \quad C_{\eta,\rho} = \frac{\int_{\mathbb{R}^n} \eta(x)|x_1|\,dx}{2\int_D \rho^2(x)\,dx}.$$

# Sketch of Proof: Convergence of Graph Total Variation

Recall

$$GTV_n(u) = \frac{1}{\epsilon_n}\frac{1}{n^2}\sum_{i,j}\eta_{\epsilon_n}(X_i - X_j)|u(X_i) - u(X_j)|.$$

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- Exponential bounds for $U$-statistics (Giné, Latała, & Zinn '00)

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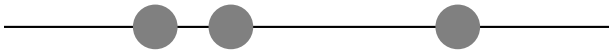

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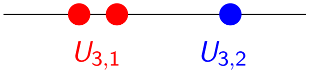

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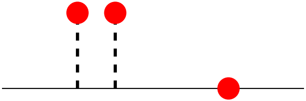

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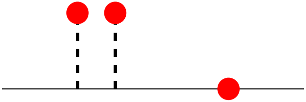

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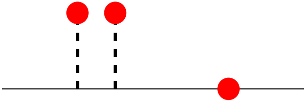

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Given partition $\mathcal{U}_n = \{U_{n,k}\}_{k=1}^K$ of $\mathcal{X}_n$, we associate the measures $\gamma_{n,k} \in \mathcal{P}(D \times \{0,1\})$, for $k = 1, \ldots, K$, by

$$\gamma_{n,k} = \frac{1}{n} \sum_{i=1}^{n} \delta_{(x_i, \mathbb{1}_{U_{n,k}}(x_i))} = (Id \times \mathbb{1}_{U_{n,k}})_\sharp \nu_n.$$

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We say that $\mathcal{U}_n \xrightarrow{w} \mathcal{U}$ if there exists a sequence $\{\pi_n\}_{n \in \mathbb{N}}$ of permutations of $\{1, \dots, K\}$ such that

$$\gamma_{n,\pi_n k} \xrightarrow{w} \gamma_k, \quad \text{for } k = 1, \dots, K.$$

# Convergence of Optimal Partitions

## Theorem (Convergence of Optimal Partitions)

*Suppose* $\{\epsilon_n\}_{n\in\mathbb{N}}$ *satisfies suitable conditions. For* $n \geq 1$*, let* $\mathcal{U}_n^* \in \arg\max_{|\mathcal{U}|\leq K} Q_n(\mathcal{U})$ *be an optimal partition.*
*If* $\mathcal{U}^*$ *is the unique solution (up to relabeling of its constituent sets) to the problem*

$$\underset{\substack{|\mathcal{U}|=K \\ \mu(U_k)=1/K}}{\text{minimize}} \sum_{k=1}^{K} \mathrm{Per}(U_k; \rho^2) \tag{P}$$

*with* $d\mu(x) = \rho^{1+\alpha}(x)\,dx / \int_D \rho^{1+\alpha}(x)\,dx$*, then* $\mathcal{U}_n^* \xrightarrow{a.s.} \mathcal{U}^*$*.*

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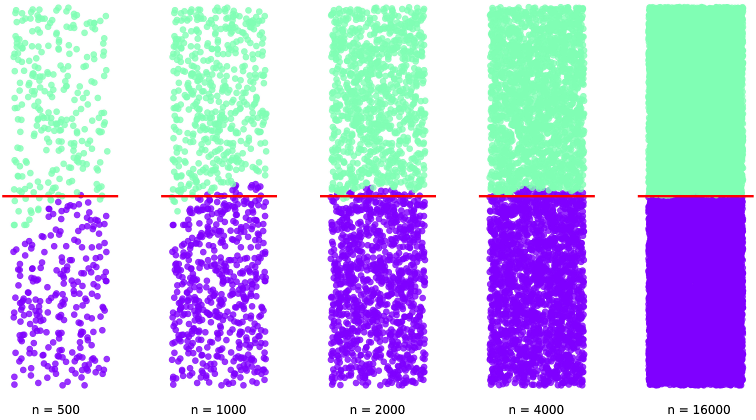
*with* $d\mu(x) = \rho^{1+\alpha}(x)\,dx / \int_D \rho^{1+\alpha}(x)\,dx$, *then* $\mathcal{U}_n^* \xrightarrow{a.s.} \mathcal{U}^*$. *If there is more than one solution to (P), then almost surely i)* $\{\mathcal{U}_n^*\}_{n\in\mathbb{N}}$ *has at least one cluster point, and ii) every cluster point is a solution to (P).*

# Example



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- Because modularity clusterings are optimizers of discrete energies, we use $\Gamma$-convergence to prove that their limit is the optimizer of a continuum energy.
- Balance constraint in the continuum problem presents a technical difficulty.
- We modified the notion of $\Gamma$-convergence for random functionals to allow the use of our pointwise convergence result.

# Existence of Transport Maps

Let $D \subset \mathbb{R}^d$ be open, connected with Lipschitz boundary. Assume $\nu = \rho(x)\,dx$ with $\rho$ continuous and bounded above/below by positive constants.

## Proposition (García-Trillos, Slepčev '14)

*There is a constant $C > 0$ such that, with probability one, there exists a sequence of transportation maps $\{T_n\}_{n\in\mathbb{N}}$, $T_{n\sharp}\nu = \nu_n$ with*

$$\limsup_{n\to\infty} \frac{n^{1/2}\|Id - T_n\|_\infty}{(2\log\log n)^{1/2}} \leq C \quad (d=1),$$

$$\limsup_{n\to\infty} \frac{n^{1/d}\|Id - T_n\|_\infty}{(\log n)^{3/4}} \leq C \quad (d=2),$$

$$\limsup_{n\to\infty} \frac{n^{1/d}\|Id - T_n\|_\infty}{(\log n)^{1/d}} \leq C \quad (d\geq 3).$$

# Assumption on $\epsilon_n$

Assume $\{\epsilon_n\}_{n\in\mathbb{N}}$ is such that $\epsilon_n > 0$ and $\epsilon_n \to 0$. For $\alpha = 0, 1$, assume that

$$\lim_{n\to\infty} \frac{\sqrt{2\log\log n}}{\sqrt{n}} \frac{1}{\epsilon_n} = 0, \qquad \text{if } d = 1$$

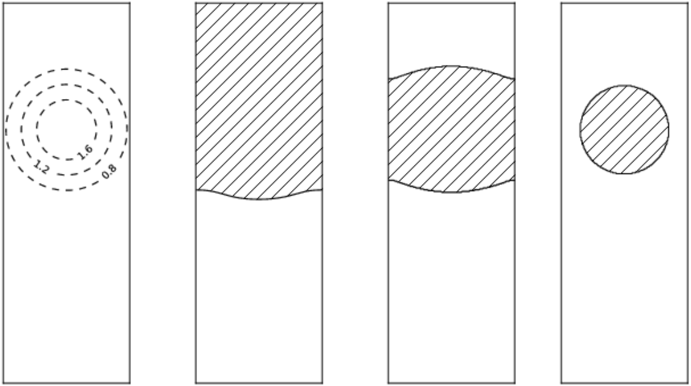
$$\lim_{n\to\infty} \frac{(\log n)^{3/4}}{n^{1/2}} \frac{1}{\epsilon_n} = 0 \qquad \text{if } d = 2$$

$$\lim_{n\to\infty} \frac{(\log n)^{1/d}}{n^{1/d}} \frac{1}{\epsilon_n} = 0 \qquad \text{if } d \geq 3.$$

For $\alpha \neq 0, 1$, assume that

$$\sum_{n=1}^{\infty} n \exp(-n\epsilon_n^{d+1}) < \infty.$$

# The role of $\alpha$




(a) Level sets (b) $\alpha = -1$ (c) $\alpha = 0$ (d) $\alpha = 1$

$$d\mu(x) \propto \rho(x)^{1+\alpha}, \ \rho(x) \propto \min(2\exp(-4||x - x_0||^2), 1/2).$$

Thanks for coming!