The Charming Leading Eigenpair

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FRONTIER PROBABILITY DAYS

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Outline

- Computing the maximal eigenpair
- Theoretic results: unified speed estimation of various stabilities
- Original motivation: study on the phase transitions

1) The maximal eigenpair

Theorem (O. Perron, 1907; G.Frobenius 1912)

For $A \geqslant 0$, irreducible, $\exists \rho(A) > 0$-maximal eigenvalue with left-eigenvector $u > 0$ and right-eigenvector $g > 0$. Dimension: one.

$$uA = \lambda u, \qquad Ag = \lambda g, \qquad \lambda = \rho(A)$$

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$$\boxed{uA = \lambda u, \qquad Ag = \lambda g, \qquad \lambda = \rho(A)}$$

Example (L.K. Hua, 1984)

$$A = \frac{1}{100}\begin{pmatrix} 25 & 14 \\ 40 & 12 \end{pmatrix},$$

$$\rho(A) = \left(37 + \sqrt{2409}\,\right)/200.$$

$u = (5(13+\sqrt{2409})/7,\ 20)$, $\boxed{\approx 44.34397483}$

$g = ((13+\sqrt{2409})/4,\ 20)$

Let $A \geqslant 0$ be irreducible and invertible.

Input-output method: $x_n = x_0 A^{-n}, \quad n \geqslant 1.$

$x_n = (x_n^{(0)}, \cdots, x_n^{(d)})$: products in nth year.

Theorem (Hua's Fundamental Theorem, 1984)

- The optimal choice is $x_0 = u$, it has the fastest grow: $x_n = x_0 \rho(A)^{-n}$.
- If $A^{-1} \not\geqslant 0$ and $x_0 \neq u$, then collapse: $\exists x_n^{(j)} \leqslant 0$.

Importance of $u \approx (44.34397483, 20)$

Table Input and collapse time

$\boldsymbol{x_0}$ [decimals]	Collapse time $\boldsymbol{n}$
$(44, 20)$	3
$(44.344, 20)$	8
$(44.34397483, 20)$	13

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$(\rho(A), u)$? High precision, large system

$$uA = \lambda u \iff A^* u^* = \lambda u^*$$

A: irreducible, off-diagonals $\geqslant 0$:

$$(A + mI)g = \lambda g \iff Ag = (\lambda - m)g$$

Main Example

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -5 & 2^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2^2 & -13 & 3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3^2 & -25 & 4^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4^2 & -41 & 5^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5^2 & -61 & 6^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6^2 & -85 & 7^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7^2 & -113 \end{pmatrix}$$

$$\rho(Q) \approx -\mathbf{0.525268}, \qquad \boxed{\rho(\text{Infinite}\, Q) = -1/4}$$

$$g \approx (55.878, 26.5271, 15.7059, 9.97983, 6.43129, 4.0251, 2.2954, 1)^*$$

Power iteration, 1929

Given $v_0 \in \mathbb{R}^{N+1}$, $v_0 \not\perp g$ with $\|v_0\| = 1$, define

$$v_k = \frac{A v_{k-1}}{\|A v_{k-1}\|}, \qquad z_k = \|A v_k\|, \qquad k \geqslant 1,$$

Then $v_k \to g$ and $z_k \to \rho(A)$ as $k \to \infty$.

$$v_k = \frac{A^k v_0}{\|A^k v_0\|} \qquad \text{“Power”}$$

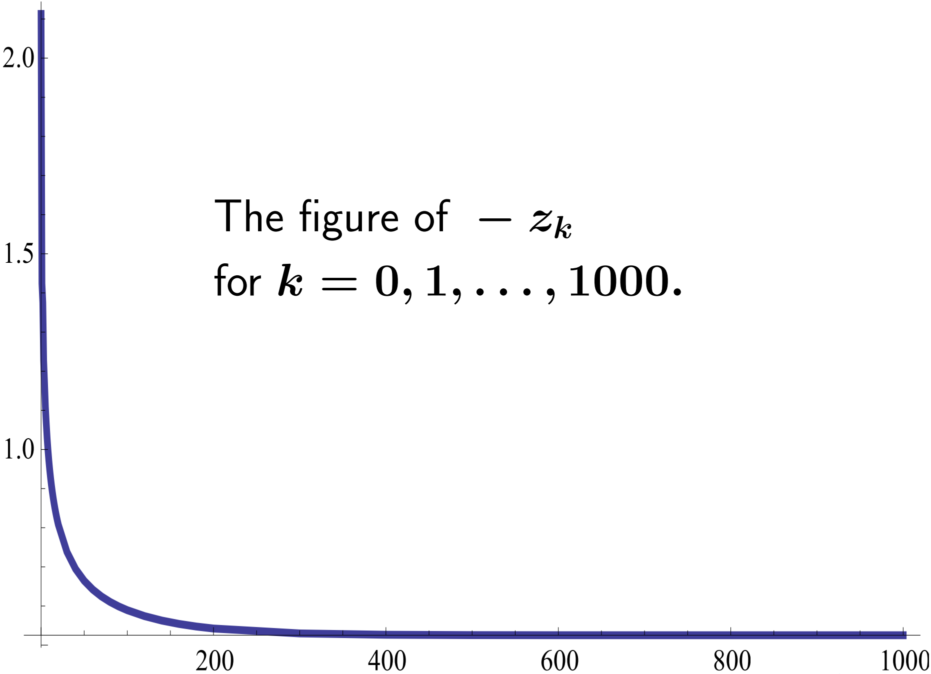
$$\tilde{v}_0 = (1,\ 0.587624,\ 0.426178,\ 0.329975,$$

$$0.260701,\ 0.204394,\ 0.153593,\ 0.101142)^*,$$

$$v_0 = \tilde{v}_0 / \|\tilde{v}_0\|, \quad \|v\| = \sum_k |v_k|, \ \ell^1\text{-norm}$$

0	**2.11289**		**11**	**0.927544**
1	**1.42407**		**12**	**0.908975**
2	**1.37537**	Computing	**13**	**0.892223**
3	**1.22712**	180 times,	**14**	**0.877012**
4	**1.1711**	10^3 iterations	**15**	**0.86311**
5	**1.10933**	64 pages	**16**	**0.850338**
6	**1.06711**		**17**	**0.838548**
7	**1.02949**		**18**	**0.827619**
8	**0.998685**	$(k, -z_k)$	**19**	**0.817449**
9	**0.971749**		**20**	**0.807953**
10	**0.948331**		**30**	**0.738257**

40	0.694746	180	0.547529
50	0.664453	200	0.542423
60	0.641946	300	0.529909
70	0.624473	400	0.526517
80	0.610468	500	0.525603
90	0.598963	600	0.525358
100	0.589332	700	0.525292
120	0.574136	800	0.525274
140	0.56279	900	0.52527
160	0.554157	$\geqslant 990$	0.525268



The figure of $-z_k$ for $\boldsymbol{k = 0, 1, \ldots, 1000.}$

Inverse iteration, 1944 ℓ^2-norm

Rayleigh quotient iteration. Choose $(z_0, v_0) \approx (\rho(A), g)$ with $v_0^* v_0 = 1$, where v^* = transpose of v. Particular, $z_0 = v_0^* A v_0$ for given v_0. At the kth step ($k \geqslant 1$), define

$$v_k = \frac{(A - z_{k-1} I)^{-1} v_{k-1}}{\|(A - z_{k-1} I)^{-1} v_{k-1}\|}, \qquad z_k = v_k^* A v_k$$

where I = identity. Then $v_k \to g$ and $z_k \to \rho(A)$ provided (z_0, v_0) is closed enough to $(\rho(A), g)$

What can we expect for the 2nd alg?

Search maximum on $(0, 1)$

with accuracy of 10^{-6}.

Golden Section Search: $10^{-6} \approx 0.618^{24}$.

Bisection Method: $10^{-6} \approx 0.5^{20}$.

Use the **Rayleigh quotient iteration**[RQI].

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Example

The same matrix Q and $\tilde{v}_0$, need 2 steps only:

$$z_1 \approx -0.528215, \quad z_2 \approx -0.525268.$$

"Too Good" is dangerous. Pitfall

$$\lambda_j := \lambda_j(-Q).\ \lambda_0 = -\rho(Q) > 0.$$

Example

Let Q be the same as above. Choose

$$v_0 = \{1, 1, 1, 1, 1, 1, 1, 1\}/\sqrt{8}.$$

Then

$\tilde{v}_0$ efficient!

$$(z_1, z_2, z_3, z_4) \approx$$

$$(4.78557,\ 5.67061,\ 5.91766,\ 5.91867).$$

$$(\lambda_0, \lambda_1, \lambda_2) \approx (0.525268,\ 2.00758,\ 5.91867).$$

Example

Let Q and v_0 be the same as in the last example. Choose

$$z_0 = 2.05768^{-1} \approx 0.485985.$$

Then $z_1 \approx 0.525313, \quad z_2 \approx 0.525268.$

z_0 efficient!

Comparison of different initials

Q	v_0	z_0	# of Iterations
1	$\tilde{v}_0$	Power	10^3
2	$\tilde{v}_0$	Auto	2
3	Uniform	Auto	Collapse
4	Uniform	δ_1^{-1}	2

- RQI is much efficient than Power One
- The initials (v_0, z_0) are very sensitive!
- It is very hard to handle with the initials

Google's PageRank

Langville, A.N., Meyer, C. D. (2006).
Google's PageRank and Beyond: The Science of Search Engine Rankings.
Princeton University Press.

Power Iteration, included.

Inverse Iteration, not touched!

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For large N, guess: # of iterations $\sim N^{\alpha}$.
$\mathbf{8^{1/3}=2}$. $\mathbf{10^{4/3} \approx 22}$. Subvert

Large N. $\lambda_0 = 1/4$ if $N = \infty$. $\leqslant 30$ Sec

Use $\tilde{v}_0$ and δ_1. Let $z_0 = 7/(8\delta_1) + v_0^*(-Q)v_0$.

$N+1$	z_0	z_1	$z_2 = \lambda_0$	upper/lower
8	0.523309	0.525268	0.525268	$1+10^{-11}$
100	0.387333	0.376393	0.376383	$1+10^{-8}$
500	0.349147	0.338342	0.338329	$1+10^{-7}$
1000	0.338027	0.327254	0.32724	$1+10^{-7}$
5000	0.319895	0.30855	0.308529	$1+10^{-7}$
7500	0.316529	0.304942	0.304918	$1+10^{-7}$
10^4	0.31437	0.302586	0.302561	$1+10^{-7}$

Efficient initials. Tridiagonal case

$$E = \{0, 1, \ldots, N\}.$$
$$Q = A - mI, \quad m := \max_{i \in E} \sum_{j \in E} a_{ij}.$$

$$Q = \begin{pmatrix} -(b_0 + c_0) & b_0 & 0 & 0 & \cdots \\ a_1 & -(a_1 + b_1 + c_1) & b_1 & 0 & \cdots \\ 0 & a_2 & -(a_2 + b_2 + c_2) & b_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & a_N & -(a_N + c_N) \end{pmatrix}$$

$$a_i > 0, \quad b_i > 0, \quad c_i \geqslant 0.$$

$$\mu_0 = 1, \ \mu_n = \mu_{n-1} \frac{b_{n-1}}{a_n}, \ n \geqslant 1.$$ Speed measure

Def $\{h_n\}$, $\{\varphi_n\}$. Explicit

$$h_0 = 1, \quad h_n = h_{n-1} r_{n-1}, \; 1 \leqslant n \leqslant N,$$

where $r_0 = 1 + c_0/b_0$,

$$r_n = 1 + \frac{a_n + c_n}{b_n} - \frac{a_n}{b_n r_{n-1}}, \; 1 \leqslant n < N,$$

$$h_{N+1} = c_N h_N + a_N (h_{N-1} - h_N),$$

$c_i \equiv 0 (i < N) \Longrightarrow h_i \equiv 1.$ $Q^{\backslash \text{the last row}} h = 0$

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$Q^{\setminus \text{the last row}} h = 0$

$$\varphi_n = \sum_{k=n}^{N} \frac{1}{h_k h_{k+1} \mu_k b_k}, \ 0 \leqslant n \leqslant N, \ b_N := 1.$$

RQ Iteration: tridiagonal case

$$\tilde{v}_0(i) = h_i\sqrt{\varphi_i}, \ i \leqslant N; \qquad v_0 = \tilde{v}_0 / \|\tilde{v}_0\|_{\mu,2};$$

$$\delta_1 = \max_{0\leqslant n\leqslant N}\left[\sqrt{\varphi_n}\sum_{k=0}^{n}\mu_k h_k^2\sqrt{\varphi_k} + \varphi_n^{-1/2}\sum_{n+1\leqslant j\leqslant N}\mu_j h_j^2\varphi_j^{3/2}\right] =: z_0^{-1}.$$

Solve w_k: $(-Q - z_{k-1}I)w_k = v_{k-1}, \quad k \geqslant 1;$

$$v_k = w_k / \|w_k\|_{\mu,2} \to g, \ z_k = (v_k, -Qv_k)_{\mu,2} \to \lambda_0$$

$(a_i, b_i, c_i) \to (\mu_i, \varphi_i, h_i)$ **Generalization**

2) Unified speed estimation of various stabilities

Theorem (Informal! 1988⟶ 2010–2014)

For tridiagonal matrix Q or one-dim elliptic operator (order 2) with/without killing on a finite/infinite interval, in each of 20 cases, there exist explicit δ, δ_1, δ_1' (and then δ_n, δ_n', recursively) such that $\delta_n' \uparrow$, $\delta_n \downarrow$ and

$$(4\delta)^{-1} \leqslant \delta_n^{-1} \leqslant \lambda_0 \leqslant {\delta_n'}^{-1} \leqslant \delta^{-1}, \qquad n \geqslant 1.$$

Besides, $1 \leqslant {\delta_1'}^{-1}/{\delta_1}^{-1} \leqslant 2$.

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One-dim elliptic operators, $\delta \longrightarrow \kappa$.

State space $E = (-M, N)$, $M, N \leqslant \infty$.

Eigenequation: $Lg = -\lambda g, \quad g \neq 0.$

Four boundaries. Use codes 'D' and 'N'.

D: (Abs.) Dirichlet boundary $g(-M) = 0$ $\boxed{\lim_M}$

N: (Ref.) Neumann boundary $g'(-M) = 0$.

- λ^{NN}: Neumann boundaries at $-M$ and N.
- λ^{DD}: Dirichlet boundaries at $-M$ and N.
- λ^{DN}: Dirichlet at $-M$ and Neumann at N.
- λ^{ND}: Neumann at $-M$ and Dirichlet at N.

Speed of L^2-exp stability $P_t = e^{tL}$

$$L = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(x)\frac{\mathrm{d}}{\mathrm{d}x}, \qquad C(x) = \int_\theta^x \frac{b}{a}$$

Speed $\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{e^C}{a}$, scale $\frac{\mathrm{d}\hat{\nu}}{\mathrm{d}x} = e^{-C}$, $\boxed{\|\cdot\| = \|\cdot\|_{L^2(\mu)}}$

$$\|P_t f\| \leqslant \|f\|\, e^{-\lambda^{\text{NN}} t}, \quad \mu(f) := \int_E f\,\mathrm{d}\mu = 0,$$

$$\|P_t f\| \leqslant \|f\|\, e^{-\lambda^{\#} t}, \quad t \geqslant 0,\ f \in L^2(\mu).$$

- $\lambda^{\text{NN}} = \lambda_1$: L^2-exponentially ergodic rate.
- Other $\lambda^{\#}$: L^2-exponential decay rate.

$\|P_t(x,\cdot) - \pi\|_{\text{Var}} \leqslant c(x) e^{-\alpha^* t}$, $P_t(x,K) \leqslant c(x,K) e^{-\alpha^* t}$. $\boxed{\alpha^* = \lambda^{\#}}$

Theorem (C. 2010). For each # of 4 cases, the

unified estimates $\left(4\kappa^{\#}\right)^{-1} \leqslant \lambda^{\#} \leqslant \left(\kappa^{\#}\right)^{-1}$,

where $\boxed{\mu(\alpha,\beta)=\int_\alpha^\beta \mathrm{d}\mu.}$

$$\left(\kappa^{\text{NN}}\right)^{-1} = \inf_{x<y} \left\{\mu(-M,x)^{-1} + \mu(y,N)^{-1}\right\} \hat{\nu}(x,y)^{-1}$$

$$\left(\kappa^{\text{DD}}\right)^{-1} = \inf_{x\leqslant y} \left\{\hat{\nu}(-M,x)^{-1} + \hat{\nu}(y,N)^{-1}\right\} \mu(x,y)^{-1}$$

$$\kappa^{\text{DN}} = \sup_{x\in(-M,N)} \hat{\nu}(-M,x)\,\mu(x,N)$$ 1920–1972

$$\kappa^{\text{ND}} = \sup_{x\in(-M,N)} \mu(-M,x)\,\hat{\nu}(x,N)$$ μ and $\hat{\nu}$, factor 4

In particular, $\lambda^{\#} > 0$ iff $\kappa^{\#} < \infty$.

1988–2010. Three probabilistic tools + 5 steps.

Intrinsic relation between 4 constants $\boldsymbol{\kappa}^{\#}$

$$\begin{array}{ccc} \boldsymbol{\kappa}^{\mathrm{DD}} & \xrightarrow{\text{Remove } \hat{\boldsymbol{\nu}}(\boldsymbol{y},\boldsymbol{N})^{-1}} & \boldsymbol{\kappa}^{\mathrm{DN}} \\ \updownarrow \textbf{Rule} & & \updownarrow \textbf{Rule} \\ \boldsymbol{\kappa}^{\mathrm{NN}} & \xrightarrow{\text{Remove } \boldsymbol{\mu}(\boldsymbol{y},\boldsymbol{N})^{-1}} & \boldsymbol{\kappa}^{\mathrm{ND}} \end{array}$$

Rule:

Codes $\boldsymbol{D} \leftrightarrow \boldsymbol{N}$

$\Leftrightarrow \boldsymbol{\mu} \leftrightarrow \hat{\boldsymbol{\nu}}$ in $\boldsymbol{\kappa}^{\#}$

$$L^c = a(x)\mathrm{d}^2/\mathrm{d}x^2 + b(x)\mathrm{d}/\mathrm{d}x - c(x),\ a > 0, c \geqslant 0$$

Poincaré-type inequalities: $(-M, N),\ M, N \leqslant \infty$

$$\lambda_c^{\#}\,\|f\|_{\mu,2}^2 \leqslant \|f'\|_{\nu,2}^2 + \|cf\|_{\mu,2}^2$$

$$= \text{if } f = g \longleftrightarrow \lambda_c^{\#}$$

$$\nu(\mathrm{d}x) = e^{C(x)}\mathrm{d}x. \quad \boxed{\hat{\nu}(\mathrm{d}x) = e^{-C(x)}\mathrm{d}x}$$

$$\sqrt{\lambda^{\#}}\,\|f\|_{\mu,2} \leqslant \|f'\|_{\nu,2} \quad (\text{when } c \equiv 0)$$

Hardy-type inequalities: $\hat{\nu}(\mathrm{d}x) = \exp\left[-\frac{C(x)}{p-1}\right]\mathrm{d}x$

$$\|f\|_{\mu,q} \leqslant A^{\#}\|f'\|_{\nu,p}. \quad \boxed{p, q \in (1, \infty).\ t^{-\alpha}}$$

$\||f|^q\|_{\mathbb{B}}^{1/q} \leqslant A_{\mathbb{B}}^{\#}\|f'\|_{\nu,p}$, New analytic proof!

Normed linear space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}}, \mu)$

$\mathbb{B}$ subset of Borel meas funs on $(X, \mathscr{X}, \mu)$.

Hypotheses (H). Norm on $\mathbb{B}$ is defined by

$$\|f\|_{\mathbb{B}} = \sup_{g \in \mathscr{G}} \int_X |f| g \,\mathrm{d}\mu,$$

where $\mathscr{G} \subset \mathscr{X}/\mathbb{R}_+$.

$\mathscr{G} = L^p (p > 1) \Longrightarrow \mathbb{B} = L^q$.

$\|f\|_{L^1(\pi)} \leqslant \mathrm{Ent}(f) \leqslant \|f\|_{L^{1+\varepsilon}(\pi)}^{1+\varepsilon}$. Interpolation.

$\mathscr{G} = \{g \geqslant 0 : \int_X e^g \mathrm{d}\pi \leqslant e^2 + 1\}$ LogSobolev

$\boxed{\text{DD: } f(-M) = f(N) = 0}$.

$\kappa^{\#} \leqslant \lambda^{\# -1} \leqslant 4\kappa^{\#}, \quad B^{\#} \leqslant A^{\#} \leqslant 2B^{\#}$ DD

$B_{\mathbb{B}}$	$\sup_{x \leqslant y} \frac{\|\mathbb{1}_{(x,y)}\|_{\mathbb{B}}^{1/q}}{\left\{\hat{\nu}(-M,x)^{1-p} + \hat{\nu}(y,N)^{1-p}\right\}^{1/p}}$
$\mathbb{B} = L^1(\mu)$ B	$\sup_{x \leqslant y} \frac{\mu(x,y)^{1/q}}{\left\{\hat{\nu}(-M,x)^{1-p} + \hat{\nu}(y,N)^{1-p}\right\}^{1/p}}$
$q = p = 2$ κ	$\sup_{x \leqslant y} \frac{\mu(x,y)}{\hat{\nu}(-M,x)^{-1} + \hat{\nu}(y,N)^{-1}}$
Killing c κ_c	$\sup_{x \leqslant y} \frac{\mu_c(x,y)}{\hat{\nu}_c(-M,x)^{-1} + \hat{\nu}_c(y,N)^{-1}}$

$h^2\mu, h^{-2}\hat{\nu}, L^c h = 0$. In parallel for ridiagonal matrix

3) Exp stability in L^2 or entropy

Semigroup $P_t = e^{tL}$. $L^2(\pi)$, $\|\cdot\|$, $(\cdot,\cdot)$.

L self-adjoint: $(f, Lg) = (Lf, g)$.

$$\|P_t f - \pi(f)\| \leqslant \|f\| e^{-\varepsilon t}, \quad \varepsilon_{\max} = \lambda_1 := \lambda^{\text{NN}}.$$

Exponential stability in entropy

$$\text{Ent}(P_t f) \leqslant \text{Ent}(f) e^{-2\sigma t}, \qquad t \geqslant 0,$$

$$\text{Ent}(f) := H(\mu \| \pi) = \int_E f \log f \,\mathrm{d}\pi \quad \text{if} \quad \frac{\mathrm{d}\mu}{\mathrm{d}\pi} = f$$

The φ^4 Euclidean quantum field on the lattice

$$\mathbb{R}^{\mathbb{Z}^d} \ni x : \mathbb{Z}^d \to \mathbb{R}. \;\; H(x) = -2J \sum_{\langle ij \rangle} x_i x_j, \;\; J, \beta \geqslant 0.$$

$$L = \sum_{i \in \mathbb{Z}^d} \left[\partial_{ii} - (u'(x_i) + \partial_i H)\partial_i\right], \; u(x_i) = x_i^4 - \beta x_i^2$$

Theorem (C. 2008)

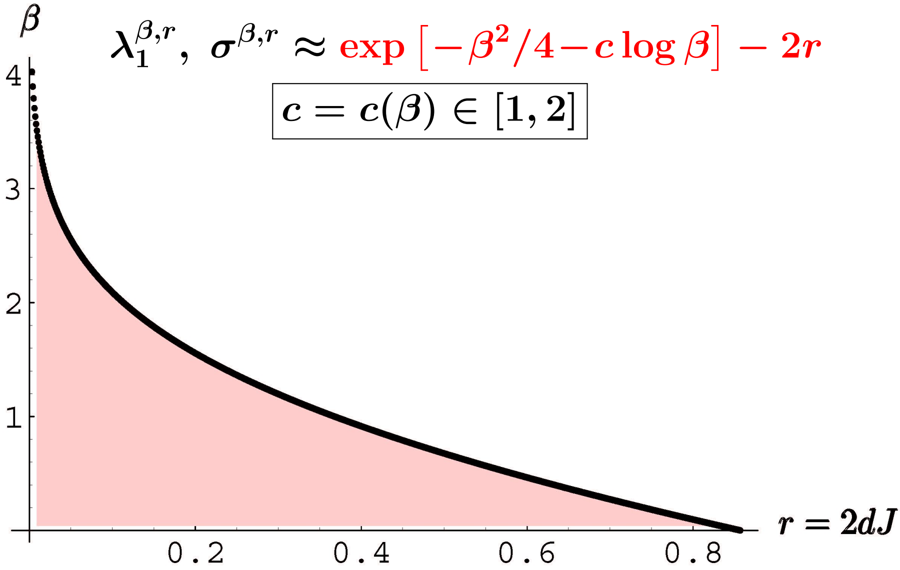
$$\inf_{\Lambda \in \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{\mathbb{Z}^d}} \lambda_1^{\beta, J}(\Lambda, \omega) \approx \inf_{\Lambda \in \mathbb{Z}^d} \inf_{\omega \in \mathbb{R}^{\mathbb{Z}^d}} \sigma^{\beta, J}(\Lambda, \omega)$$

$$\approx \exp\left[-\beta^2/4 - c \log \beta\right] - 4dJ \quad \boxed{c \in [1, 2]}$$

Phase transition: the φ^4 model

$$\lambda_1^{\beta,r},\ \sigma^{\beta,r} \approx \exp\left[-\beta^2/4 - c\log\beta\right] - 2r$$

$$\boxed{c = c(\beta) \in [1, 2]}$$



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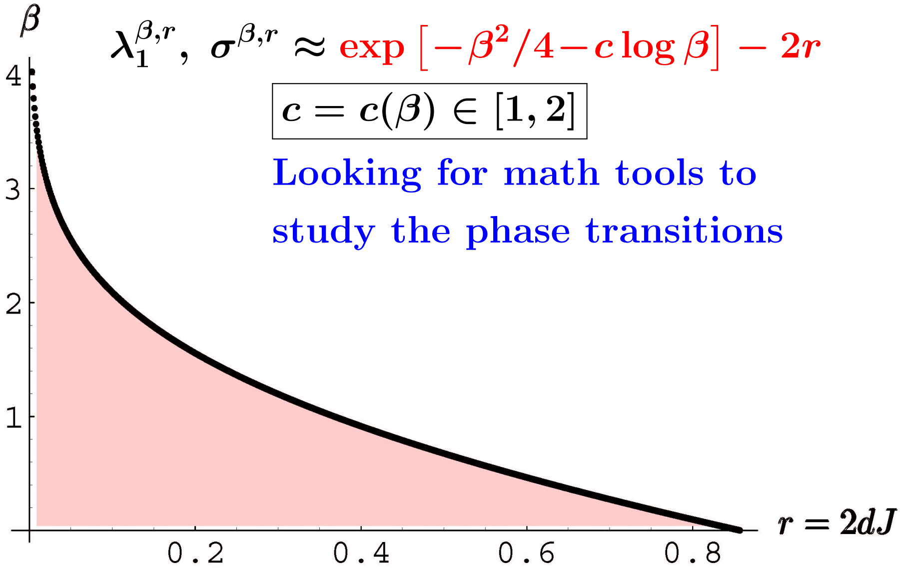
Looking for math tools to study the phase transitions



For Further Reading I

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Spectral gap and logarithmic Sobolev constant for continuous spin systems
Acta Math. Sin. New Ser. 24(5): 705–736

C. (2010)
Speed of stability for birth–death processes
Front. Math. China. 5:3, 379–515

For Further Reading II

- C. (2014)
  Criteria for discrete spectrum of 1D operators
  Commun. Math. Stat. 2014, 2(3): 279 – 309

- C. (2016) Survey article, free downloadable!
  Unified speed estimation of various stabilities
  Chin. J. Appl. Probab. Statis. 32(1), 1–22

- C. (2016): *Efficient initials for computing the maximal eigenpair*, preprint

For Further Reading III

- C. and Zhang, X. (2014).
  Isospectral operators.
  Commu. Math. Stat. 2, 17–32
- C. and Zhang, Y.H. (2014).
  Unified representation of formulas for single birth processes.
  Front. Math. China 9(4): 761–796

http://math0.bnu.edu.cn/~chenmf 4 volumes

The end!

Thank you, everybody!

谢谢大家!

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