Limit Laws for Random Matrices from Traffic Probability

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Wigner's Semicircle Law

Definition (Wigner matrix)

Let $(W_{i,j})_{1 \leq i < j < \infty}$ and $(W_{i,i})_{1 \leq i < \infty}$ be independent families of i.i.d. real-valued random variables such that $\mathbb{E}W_{1,2} = 0$, $\mathrm{Var}(W_{1,2}) = 1$, and $\mathrm{Var}(W_{1,1}) < \infty$. We call the random real symmetric $n \times n$ matrix $\mathbf{W}_n$ defined by

$$\mathbf{W}_n(i,j) = \begin{cases} W_{i,j}/\sqrt{n} & \text{if } i < j \\ W_{i,i}/\sqrt{n} & \text{if } i = j \end{cases}$$

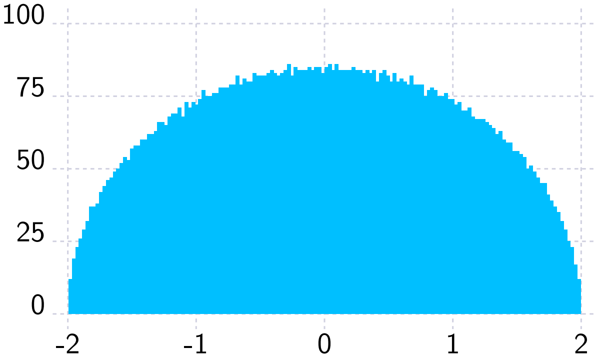
a *Wigner matrix*.

Theorem (Wigner, 1955)

The empirical spectral distributions (ESDs) $\mu(\mathbf{W}_n)$ converge weakly almost surely to the standard semicircle distribution $\mathcal{SC}(0,1)$, where

$$\mathcal{SC}(m,\sigma^2)(dx) = \frac{1}{2\pi\sigma^2}(4\sigma^2 - (x-m)^2)_+^{1/2}\, dx.$$

Wigner's Semicircle Law



Consider first the case of a usual measurable space $(\Omega, \mathcal{F})$. For a given probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, we may form the (commutative) $*$-algebra $L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})$ of measurable complex-valued functions having finite moments of all orders, i.e.,

$$L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}) = \bigcap_{p=1}^{\infty} L^p(\Omega, \mathcal{F}, \mathbb{P}).$$

The expectation $\mathbb{E} : L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{C}$ recovers the probability measure $\mathbb{P}$; thus, the passage from the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ((\Omega, \mathcal{F}), \mathbb{P})$ to the pair $(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ involves no loss of information.

Non-commutative Probability

Definition (Non-commutative probability space)

A *non-commutative probability space* is a pair $(\mathcal{A}, \varphi)$ consisting of a unital algebra $\mathcal{A}$ over $\mathbb{C}$ equipped with a unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$.

Examples

- $(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$
- $(M_n(\mathbb{C}), \frac{1}{n}\mathrm{tr})$
- $(M_n(L^{\infty-}(\Omega, \mathcal{F}, \mathbb{P})), \mathbb{E}\frac{1}{n}\mathrm{tr})$
- $(\mathbb{C}[G], \tau_G)$

Notions of Independence

For a collection of random variables $\mathcal{S} \subset \mathcal{A}$, we write $\mathring{\mathcal{S}}$ for the subset (possibly empty) of centered random variables in $\mathcal{S}$.

Definition (Classical independence)

Subalgebras $(\mathcal{A}_i)_{i \in I}$ are *classically independent* if the $\mathcal{A}_i$ commute and φ is multiplicative across the $\mathcal{A}_i$ in the following sense: for all $k \geq 1$ and distinct indices $i(1), \ldots, i(k) \in I$,

$$\varphi\left(\prod_{j=1}^{k} a_{i(j)}\right) = \prod_{j=1}^{k} \varphi(a_{i(j)}), \quad \forall a_{i(j)} \in \mathcal{A}_{i(j)}, \tag{1}$$

or, equivalently,

$$\varphi\left(\prod_{j=1}^{k} a_{i(j)}\right) = 0, \quad \forall a_{i(j)} \in \mathring{\mathcal{A}}_{i(j)}. \tag{2}$$

Notions of Independence

Voiculescu: "What is free probability theory? It is not a euphemism for the advocacy of an unconstrained attitude in the practice of probability. It can rather be described by the exact formula

free probability = non-commutative probability theory+free independence."

Definition (Free independence)

Subalgebras $(\mathcal{A}_i)_{i\in I}$ are *freely independent* if for all $k \geq 1$ and consecutively distinct indices $i(1) \neq i(2) \neq \cdots \neq i(k) \in I$,

$$\varphi\Big(\prod_{j=1}^{k} a_{i(j)}\Big) = 0, \quad \forall a_{i(j)} \in \mathring{\mathcal{A}}_{i(j)}. \tag{3}$$

Non-commutative Central Limit Theorems (CLTs)

Theorem (de Moivre, 1733; Voiculescu, 1985)

Let (a_n) be a sequence of identically distributed self-adjoint random variables in a $*$-probability space $(\mathcal{A}, \varphi)$. Assume that the a_n are centered with unit variance, i.e., $\varphi(a_n) = 0$ with $\varphi(a_n^2) = 1$, and write $s_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n a_j$. We consider two cases.

(i) *If the a_n are classically independent, then (s_n) converges in distribution to a standard normal random variable, i.e.,*

$$\lim_{n\to\infty} \varphi(s_n^m) = \int_{\mathbb{R}} t^m \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt, \qquad \forall m \in \mathbb{N}.$$

(ii) *If the a_n are freely independent, then (s_n) converges in distribution to a standard semicircular random variable, i.e.,*

$$\lim_{n\to\infty} \varphi(s_n^m) = \int_{-2}^{2} t^m \cdot \frac{1}{2\pi} \sqrt{4 - t^2}\, dt, \qquad \forall m \in \mathbb{N}.$$

Limit Laws for Random Matrices from Free Probability

Theorem (Voiculescu, 1991; Dykema, 1993)

With the appropriate moment assumptions, independent Wigner matrices are asymptotically freely independent.

Corollary

The ESDs $\mu(\mathbf{W}_n)$ converge in expectation to the standard semicircle distribution $\mathcal{SC}(0,1)$.

Random Markov Matrices

Definition (Markov matrix)

Let $\mathbf{W}_n$ be a Wigner matrix, and let $\mathbf{D}_n$ be the diagonal matrix of row sums of $\mathbf{W}_n$, i.e.,

$$\mathbf{D}_n(i,i) = \sum_{j=1}^{n} \mathbf{W}_n(i,j) = \sum_{j=1}^{n} W_{i,j}/\sqrt{n}.$$

We call the random real symmetric $n \times n$ matrix $\mathbf{M}_n$ defined by

$$\mathbf{M}_n = \mathbf{W}_n - \mathbf{D}_n$$

a *Markov matrix*.

Theorem (Bryc, Dembo, and Jiang, 2006)

The ESDs $\mu(\mathbf{M}_n)$ converge weakly almost surely to the free convolution $\mathcal{N}(0,1) \boxplus \mathcal{SC}(0,1)$.

$$\mathbf{M}_n = \frac{1}{\sqrt{n}} \begin{pmatrix} -\sum_{j\neq 1}^{n} W_{1,j} & W_{1,2} & W_{1,3} & \cdots & & W_{1,n} \\ W_{2,1} & -\sum_{j\neq 2}^{n} W_{2,j} & W_{2,3} & \cdots & & W_{2,n} \\ W_{3,1} & W_{3,2} & \ddots & & & \vdots \\ \vdots & \vdots & & & & \\ W_{k,1} & W_{k,2} & \cdots & -\sum_{j\neq k}^{n} W_{k,j} & \cdots & W_{k,n} \\ \vdots & \vdots & & & \ddots & \vdots \\ W_{n,1} & W_{n,2} & \cdots & & & -\sum_{j\neq n}^{n} W_{n,j} \end{pmatrix}$$

Generalized Notions of Independence

Do there exist other notions of independence in the non-commutative probabilistic setting?

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Theorem (Speicher, 1997)

No.

$*$-graph polynomials

We write $\mathbf{x} = (x_i)_{i \in I}$ for a set of indeterminates. We implicitly assume an associated set of indeterminates $\mathbf{x}^* = (x_i^*)_{i \in I}$ such that

$$(\mathbf{x}, \mathbf{x}^*) = (x_i, x_i^*)_{i \in I}$$

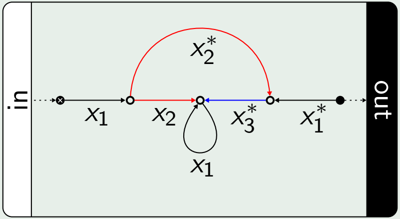
gives a set of pairwise distinct indeterminates satisfying the natural $*$-relation.

Definition ($*$-graph monomial)

A $*$-*graph monomial* in the indeterminates $\mathbf{x}$ is a finite, connected, bi-rooted $*$-graph in $\mathbf{x}$. We denote the set of $*$-graph monomials in $\mathbf{x}$ by $\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$.

$*$-graph polynomials

Example



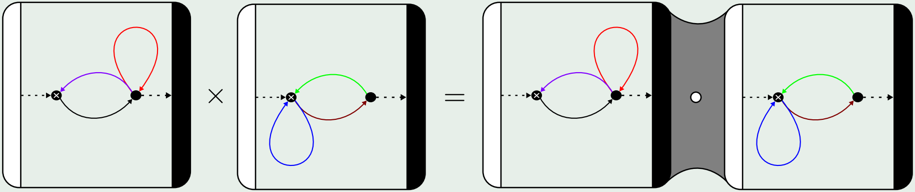
$*$-graph polynomials

Definition ($*$-graph polynomials)

We write $\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$ for the complex vector space of finite linear combinations in $\mathcal{G}\langle \mathbf{x}, \mathbf{x} \rangle$, the elements of which we call the *$*$-graph polynomials*.

$*$-graph polynomials

Example



$*$-graph polynomials

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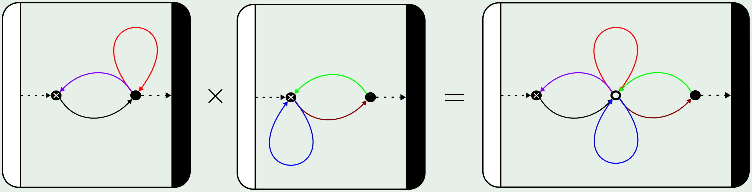

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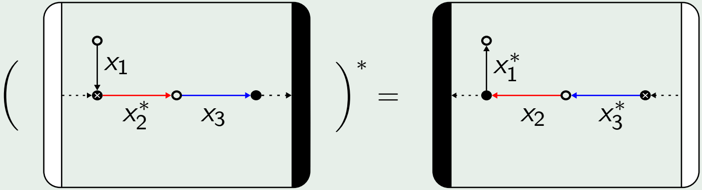
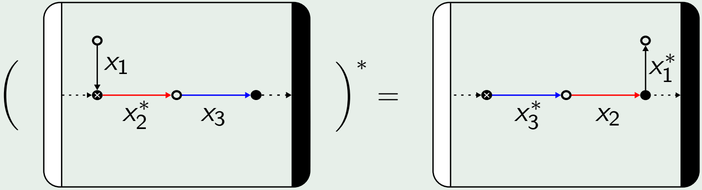


$*$-graph polynomials

Example



Definition (Traffic space)

A *traffic space* is a tracial $*$-probability space $(\mathcal{A}, \varphi)$ such that $\mathcal{A}$ is an algebra over the symmetric operad of $*$-graph polynomials.

Traffic Probability

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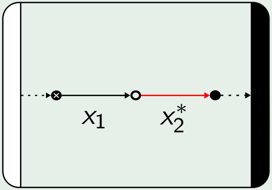
Example

Let $\mathbf{A} = (A_i)_{i \in I}$ be a family of random $n \times n$ matrices. For a $*$-graph monomial $t = (T, v_{\text{in}}, v_{\text{out}}) = (V, E, f, g, \gamma, \varepsilon, v_{\text{in}}, v_{\text{out}})$ in $\mathbf{x} = (x_i)_{i \in I}$, we define $t(\mathbf{A})$ to be the random $n \times n$ matrix with entries

$$t(\mathbf{A})(i,j) = \sum_{\substack{\phi : V \to [n] \\ \phi(v_{\text{in}}) = i,\, \phi(v_{\text{out}}) = j}} \prod_{e \in E} A_{\gamma(e)}^{\varepsilon(e)}(\phi(f(e)), \phi(g(e)))$$

Traffic Probability

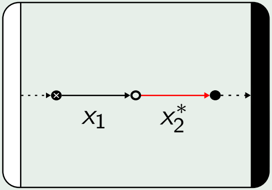
Example



$$\left(A_1, A_2\right)$$

Traffic Probability

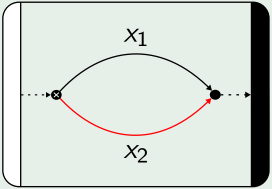
Example



$$\Big(A_1, A_2\Big) = A_1 A_2^*$$

Traffic Probability

Example



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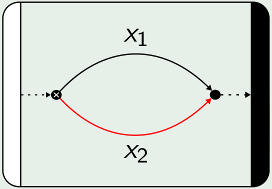
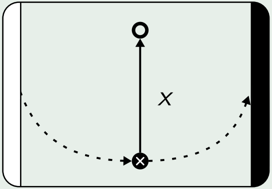


$$\left(A_1, A_2\right) = A_1 \circ A_2$$

Traffic Probability

Example



$(A) = \mathrm{row}(A)$

CLTs Revisited

Theorem (de Moivre, 1733; Voiculescu, 1985)

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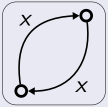
CLTs Revisited

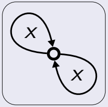
Theorem (Male, 2012)

(iii) *Assume that $\mathcal{A}$ has the additional structure of a traffic space $(\mathcal{A}, \varphi, \tau)$. We split the variance of a_n as*

$$1 = \varphi(a_n^2) = \tau\big[T_1(a_n)\big] = \tau^0\big[T_1(a_n)\big] + \tau^0\big[T_2(a_n)\big] = \alpha + (1-\alpha),$$

where

$T_1 =$ 

and $T_2 =$ 

.

If the a_n are traffic independent, then (s_n) converges in distribution to the free convolution $\mu_\alpha = \mathcal{SC}(0,\alpha) \boxplus \mathcal{N}(0, 1-\alpha)$, i.e.,

$$\lim_{n\to\infty} \varphi(s_n^m) = \int_{\mathbb{R}} t^m \, \mu_\alpha(dt), \qquad \forall m \in \mathbb{N}.$$

Limit Laws for Random Matrices from Traffic Probability

Definition ((p, q)-Markov matrices)

Let $\mathbf{W}_n$ be a Wigner matrix and $\mathbf{D}_n = \mathrm{row}(\mathbf{W}_n)$ the diagonal matrix of row sums of $\mathbf{W}_n$. For $p, q \in \mathbb{R}$, we call the real symmetric $n \times n$ matrix $\mathbf{M}_{n,p,q}$ defined by

$$\mathbf{M}_{n,p,q} = p\mathbf{W}_n + q\mathbf{D}_n$$

a (p, q)-*Markov matrix*.

Theorem (Au, 2016)

Let $(\mathbf{W}_n^{(\ell)} : 1 \leq \ell < \infty)$ be a sequence of independent finite moments Wigner matrices. Then the families $((\mathbf{M}_{n,p,q}^{(\ell)})_{p,q\in\mathbb{R}} : 1 \leq \ell < \infty)$ are asymptotically traffic independent with stable universal limiting traffic distribution.

Corollary

The ESDs $\mu(\mathbf{M}_{n,p,q})$ *converge in expectation to the free convolution* $\mathcal{SC}(0, p^2) \boxplus \mathcal{N}(0, q^2)$.

Limit Laws for Random Matrices from Traffic Probability

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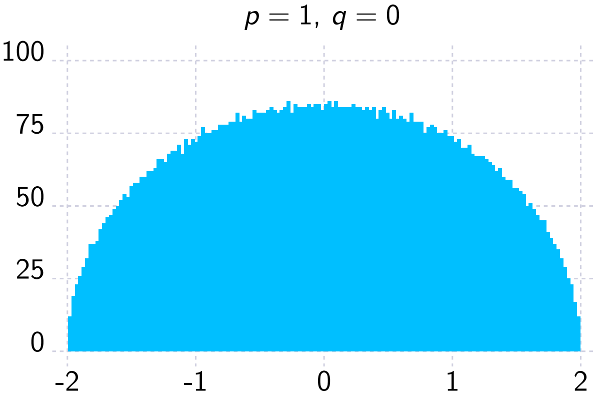
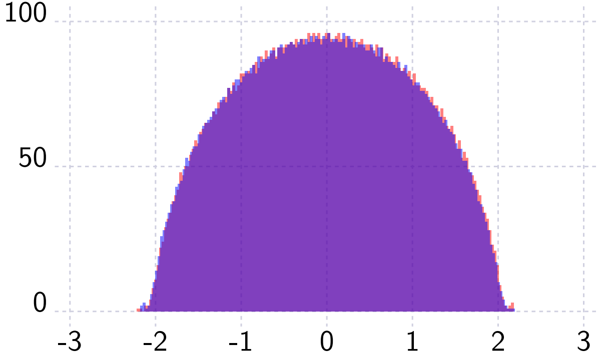


Corollary

The ESDs $\mu(\mathbf{M}_{n,p,q})$ converge in expectation to the free convolution $\mathcal{SC}(0, p^2) \boxplus \mathcal{N}(0, q^2)$.

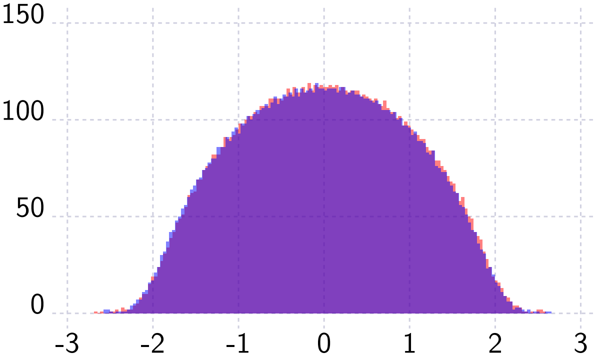
$p = \sqrt{.8}$, $q = \pm\sqrt{.2}$



Corollary

The ESDs $\mu(\mathbf{M}_{n,p,q})$ converge in expectation to the free convolution $\mathcal{SC}(0, p^2) \boxplus \mathcal{N}(0, q^2)$.

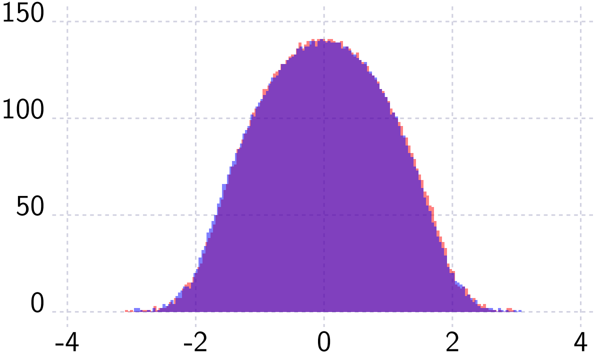
$p = \sqrt{.6}$, $q = \pm\sqrt{.4}$



Corollary

The ESDs $\mu(\mathbf{M}_{n,p,q})$ converge in expectation to the free convolution $\mathcal{SC}(0, p^2) \boxplus \mathcal{N}(0, q^2)$.

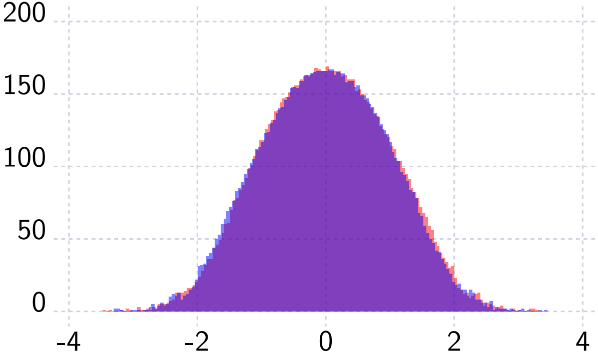
$p = \sqrt{.4}$, $q = \pm\sqrt{.6}$



Corollary

The ESDs $\mu(\mathbf{M}_{n,p,q})$ converge in expectation to the free convolution $\mathcal{SC}(0, p^2) \boxplus \mathcal{N}(0, q^2)$.

$p = \sqrt{.2}$, $q = \pm\sqrt{.8}$



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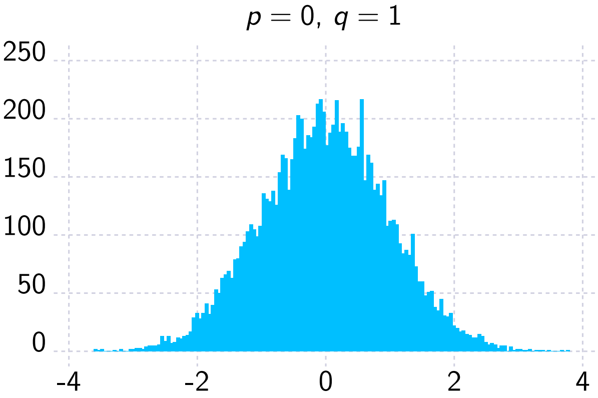

Theorem (Au, 2016)

Let $\mathbf{W}_n$ be a Wigner matrix, and let (X_n) and (Y_n) be sequences of real-valued random variables converging almost surely to X and Y respectively. Then the ESDs $\mu(\mathbf{M}_{n,X_n,Y_n})$ converge weakly almost surely to the random free convolution $\mathcal{SC}(0, X^2) \boxplus \mathcal{N}(0, Y^2)$.

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Thank you!