An Integro-Differential Equation Driven by Fractional Brownian Motion

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Fractional Brownian Motion (FBM)

Let $B = \{B_t, t \geq 0\}$ be a real valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$, and $H \in (0,1)$. B is called a FBM with Hurst parameter H, if it is a centered Gaussian process with the covariance function

$$R_H(s,t) = E[B^H(s)B^H(t)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \quad (1)$$

A stochastic process $\{X_t, 0 \leq t \leq T\}$ is β Hölder continuous $(0 < \beta < 1)$ if there exists an a.s. finite random variable $K(\omega)$ such that

$$\sup_{s,t \in [0,T], s \neq t} \frac{|X_t - X_s|}{|t-s|^{\beta}} \leq K(\omega) \quad (2)$$

FBM admits a modification which is Hölder continuous of order β iff $\beta \in (0, H)$.

Some Normed Spaces

Let $\frac{1}{2} \leq H \leq 1$, $1 - H \leq \alpha \leq \frac{1}{2}$.

Let $C([0,T], W^{\alpha,\infty}[0,1])$ be the space of measurable functions $f : [0,T] \times [0,1] \to \mathbb{R}$ such that

$$\| f \|_{\alpha,\infty} := \sup_{t\in[0,T]} \sup_{\xi\in[0,1]} \left(|f(t,\xi)| + \int_0^{\xi} \frac{|f(t,\xi) - f(t,\eta)|}{|\xi - \eta|^{\alpha+1}} d\eta \right) < \infty. \qquad (3)$$

Let $C([0,T], W^{1-\alpha,\infty,0}[0,1])$ be the space of measurable functions $g : [0,T] \times [0,1] \to \mathbb{R}$ such that

$$\| g \|_{1-\alpha,\infty,0} := \sup_{t\in[0,T]} \sup_{0<\eta<\xi<1} \left(\frac{|g(t,\xi) - g(t,\eta)|}{(\xi - \eta)^{1-\alpha}} + \int_{\eta}^{\xi} \frac{|g(t,\gamma) - g(t,\eta)|}{(\gamma - \eta)^{2-\alpha}} d\gamma \right) < \infty. \qquad (4)$$

Let $W^{\alpha,1}([0,1])$ be the space of measurable functions $f : [0,1] \to \mathbb{R}$ such that

$$\| f \|_{\alpha,1} := \int_0^1 \frac{|f(\eta)|}{\eta^{\alpha}} d\eta + \int_0^1 \int_0^{\eta} \frac{|f(\eta) - f(\delta)|}{|\eta - \delta|^{\alpha+1}} d\delta d\eta < \infty. \qquad (5)$$

Generalized Stieltjes Integrals

Define the following quantities assuming the limits exist and are finite. Let
$f(a+) = \lim_{\epsilon \searrow 0} f(a+\epsilon)$,
$g(b-) = \lim_{\epsilon \searrow 0} g(b-\epsilon)$,
$f_{a+}(x) = [f(x) - f(a+)]1_{(a,b)}(x)$ and
$g_{b-}(x) = [g(x) - g(b-)]1_{(a,b)}(x)$.

Definition

Suppose that f and g are functions such that $f(a+)$, $g(a+)$ and $g(b-)$ exist, $f_{a+} \in I_{a+}^{\alpha}(L^p)$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^q)$ for some $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} \leq 1$, $0 < \alpha < 1$. If $\alpha p < 1$ then we have $f \in I_{a+}^{\alpha}(L^p)$. Then integral of f with respect to g can be defined as follows;

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x) dx. \tag{6}$$

A Priori Estimate

Let

$$\Lambda_\alpha(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0<\eta<\xi<1} |(D^{1-\alpha}_{\xi-} g_{\xi-})(\eta)|. \tag{7}$$

Then

$$\Lambda_\alpha(g) \leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \parallel g \parallel_{1-\alpha,\infty,0} < \infty. \tag{8}$$

If $f \in W^{\alpha,1}(0,1)$, and $g \in W^{1-\alpha,\infty,0}(0,1)$ then the integral $\int_0^\xi fdg$ exists for all $\xi \in [0,1]$.

Using (6) we get

$$\left| \int_0^\xi fdg \right| \leq \sup_{0<\eta<\xi} |(D^{1-\alpha}_{\xi-} g_{\xi-})(\eta)| \int_0^\xi |(D^\alpha_{0+} f)(\eta)| d\eta.$$

Hence

$$\left| \int_0^\xi fdg \right| \leq \Lambda_\alpha(g) \parallel f \parallel_{\alpha,1} . \tag{9}$$

Stochastic Integral

If $1 - H < \alpha < \frac{1}{2}$ then,

$$\mathbb{E} \sup_{0 \le s \le t \le T} |D_{t-}^{1-\alpha} B_{t-}^H(s)|^p < \infty, \tag{10}$$

for all $T > 0$ and $p \ge 1$, and we have

$$\| B_t^H - B_s^H \|_p = [\mathbb{E}(|B_t^H - B_s^H|^p)]^{\frac{1}{p}} = c_p |t-s|^H. \tag{11}$$

We know that the random variable

$$G = \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |D_{t-}^{1-\alpha} B_{t-}^H(s)| \tag{12}$$

has moments of all orders.
As a consequence for $1 - H < \alpha < \frac{1}{2}$ the pathwise integral $\int_0^t u_s dB_s^H$ exists where $u = \{u_t, t \in [0, T]\}$ is a stochastic process whose trajectories belong to the space $W_T^{\alpha,1}$ and B is a FBM with $H > \frac{1}{2}$. Moreover, we have the estimate

$$\left| \int_0^T u_s dB_s^H \right| \le G \|u\|_{\alpha,1}. \tag{13}$$

Background

we will assume that the vorticity field associated to an ideal inviscid incompressible homogeneous fluid in $\mathbb{R}^3$ is described by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ and we will study its evolution through a pathline equation. Let us denote by $\vec{\omega} \in \mathbb{R}^3$ this vorticity field and for $\vec{u} \in \mathbb{R}^3$ we have $\vec{\omega} := \nabla \times \vec{u}$.

$$\frac{d\vec{X}_t(\vec{x})}{dt} = \vec{u}(t, \vec{X}_t(\vec{x})) \tag{14}$$

$$\vec{\omega}(\vec{x}, t) = \int_0^1 \delta(\vec{x} - \vec{B^H}(t, \xi)) d\vec{B^H}(t, \xi) \tag{15}$$

$$\frac{\partial \vec{X}(t, \xi)}{\partial t} = \int_0^1 Q(\vec{X}(t, \xi) - \vec{B^H}(t, \eta)) d\vec{B^H}(t, \eta) \tag{16}$$

with

$$\vec{X}(0, \xi) = \vec{\psi}(\xi)$$

Equation Studied

We are interested in the following integro-differential equation

$$Y(t,\xi) = \phi(\xi) + \int_0^t \int_0^\xi A(Y(s,\eta))dB^H(s,\eta)ds, \tag{17}$$

where B^H is a real valued fractional Brownian motion ($H > \frac{1}{2}$) defined on a complete probability space $(\Omega, \mathcal{F}, P)$, $\phi(\xi)$ is the initial condition, $\xi \in [0,1]$ is a parameter, $t \in [0, T]$ is time and $A : \mathbb{R} \to \mathbb{R}$ is a measurable function that satisfies the conditions given below.

A1. A is differentiable.
A2. There exists $M_1 > 0$ such that $|A(x) - A(y)| \leq M_1|x - y|$ for all $x, y \in \mathbb{R}$.
A3. There exists $M_2 > 0$ such that $|A(x)| \leq M_2$ for all $x \in \mathbb{R}$.
A4. There exists $0 < \zeta \leq 1$ and for every N there exists $M_N > 0$, such that $|A'(x) - A'(y)| \leq M_N|x - y|^\zeta$ for all $|x|, |y| \leq N$.

Main Result

Theorem

Let $\alpha_0 = \min\{\frac{1}{2}, \zeta\}$ *and* $\alpha \in (1 - H, \alpha_0)$. *If* $\phi \in W^{\alpha,\infty}[0,1]$ *and the function A satisfies the assumptions A1, A2, A3, and A4 then, there exists a unique stochastic process* $Y \in L^0((\Omega, \mathcal{F}, P), C([0,T], W^{\alpha,\infty}[0,1]))$ *which is a global solution of the stochastic partial differential equation* (17).

Proof - Outline

Step 1: Fix ω - work pathwise

Step 2: Local existence and uniqueness - contraction principle

Step 3: Invariance (Lemma 1)

Step 4: Contraction (Proposition 1 + Proposition 2 $\Rightarrow$ Lemma 2)
Lemma 1 + Lemma 2 $\Rightarrow$ Theorem 1 (Local existence and uniqueness)

Step 5: Global existence (Theorem 2)

Step 6: Stochastic case (Theorem)

Lemma 1 (Invariance)

Consider the operator

$$F : C([0,T], W^{\alpha,\infty}[0,1]) \to C([0,T], W^{\alpha,\infty}[0,1])$$

defined by

$$F(Y(t,\xi)) := \phi(\xi) + \int_0^t \int_0^\xi A(Y(s,\gamma))dg_\gamma ds, \tag{18}$$

where $Y \in (C([0,T], W^{\alpha,\infty}[0,1]))$, $g \in C([0,T], W^{1-\alpha,\infty,0}[0,1])$, $\phi \in W^{\alpha,\infty}[0,1]$ and A satisfies the assumptions A1, A2, A3, and A4.
Let us also define the ball

$$B_{R_1} = \{Y \in C([0,T_1], W^{\alpha,\infty}[0,1]) : ||Y||_{\alpha,\infty} \le R_1\}.$$

Lemma

Given a $R_1 > \phi(\xi)$ there exists $T_1 > 0$ such that $F(B_{R_1}) \subset B_{R_1}$. The time T_1 depends on R_1, initial condition $\phi(\xi)$ and the constant α.

Sketch of the Proof of Lemma 1

$$||F(Y(t,\xi))||_{\alpha,\infty} = \sup_{t\in[0,T],\xi\in[0,1]} \left(|F(y(t,\xi))| + \int_0^{\xi} \frac{|F(Y(t,\xi)) - F(Y(t,\eta))|}{(\xi-\eta)^{\alpha+1}} d\eta \right) \quad (19)$$

Consider the first part.

$$\begin{aligned} |F(Y(t,\xi))| &= |\phi(\xi) + \int_0^t \int_0^{\xi} A(Y(s,\gamma)) dg_{\gamma} ds| \\ &\leq |\phi(\xi)| + \int_0^t \left| \int_0^{\xi} A(Y(s,\gamma)) dg_{\gamma} ds \right| \\ &\leq |\phi(\xi)| + \Lambda_{\alpha}(g) t ||A||_{\alpha,1} \end{aligned}$$

where $\Lambda_{\alpha}(g) = \frac{1}{\Gamma(1-\alpha)} \sup_{0<\eta<\xi<1} |D^{1-\alpha}_{\xi-} g_{\xi-}(\gamma)|$.

Sketch of the Proof of Lemma 1 (cont.)

Further,

$$||A(Y(s,\gamma))||_{\alpha,1} \leq \frac{M_2}{1-\alpha} + M_1||Y||_{\alpha,\infty}.$$

Thus we get,

$$|F(Y(t,\xi))| \leq |\phi(\xi)| + \Lambda_\alpha(g)t\left(\frac{M_2}{1-\alpha} + M_1||Y||_{\alpha,\infty}\right). \tag{20}$$

Sketch of the Proof of Lemma 1 (cont.)

Now we consider the second part in (19).

$$\int_0^\xi \frac{|F(Y(t,\xi)) - F(Y(t,\eta))|}{(\xi-\eta)^{\alpha+1}} d\eta$$
$$\leq \int_0^\xi \frac{|\phi(\xi)-\phi(\eta)|}{(\xi-\eta)^{\alpha+1}} + \int_0^t \left(\int_0^\xi \frac{1}{(\xi-\eta)^{\alpha+1}} \left| \int_\eta^\xi A(Y(s,\gamma)) dg_\gamma \right| d\eta \right) ds \quad (21)$$

Now we consider the second part in (21).

$$\int_0^\xi \frac{1}{(\xi-\eta)^{\alpha+1}} \left| \int_\eta^\xi A(Y(s,\gamma)) dg_\gamma ds \right| d\eta$$
$$\leq \Lambda_\alpha(g) \int_0^\xi \frac{1}{(\xi-\eta)^{\alpha+1}} \int_\eta^\xi \frac{|A(Y(s,\gamma))|}{(\xi-\eta)^\alpha} d\gamma d\eta$$
$$+ \Lambda_\alpha(g) M_1 \int_0^\xi \frac{1}{(\xi-\eta)^{\alpha+1}} \int_\eta^\xi \int_\eta^\gamma \frac{|Y(s,\gamma)-Y(s,\delta)|}{(\gamma-\delta)^{\alpha+1}} d\delta d\gamma d\eta \qquad (22)$$

Sketch of the Proof of Lemma 1 (cont.)

Consider the first part in (22) with the substitution $\eta = \gamma - (\xi - \gamma)x$.

$$\int_0^\xi \frac{1}{(\xi-\eta)^{\alpha+1}} \int_\eta^\xi \frac{|A(Y(s,\gamma))|}{(\xi-\eta)^\alpha} d\gamma d\eta \leq \frac{M_2 b_\alpha^{(1)}}{1-2\alpha} \xi^{1-2\alpha} \tag{23}$$

where $b_\alpha^{(1)} = B(2\alpha, 1-\alpha)$ with,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 x^{p-1}(1-x)^{q-1} dx = \int_0^\infty \frac{x^{q-1}}{(1+x)^{p+q}} dx \tag{24}$$

Now consider the second part in (22).

$$\int_0^\xi \frac{1}{(\xi-\eta)^{\alpha+1}} \int_\eta^\xi \int_\eta^\gamma \frac{|Y(s,\gamma) - Y(s,\delta)|}{(\gamma-\delta)^{\alpha+1}} d\delta d\gamma d\eta \leq \frac{||Y||_{\alpha,\infty}}{\alpha(1-\alpha)} \xi^{1-\alpha} \tag{25}$$

Sketch of the Proof of Lemma 1 (cont.)

Now using the estimates (20), (21), (22), (23) and (25) with (19) we get;

$$||F(Y)||_{\alpha,\infty} \leq ||\phi||_{\alpha,\infty} + Tb_\alpha^{(2)}(1 + ||Y||_{\alpha,\infty})$$

where

$$b_\alpha^{(2)} = \left(M_2 \left[\frac{1}{1-\alpha} + \frac{b_\alpha^{(1)}}{1-2\alpha} \right] + M_1 \left[1 + \frac{1}{\alpha(1-\alpha)} \right] \right) \Lambda_\alpha(g). \qquad (26)$$

Choose $T_1 = \frac{R_1 - ||\phi||_{\alpha,\infty}}{b_\alpha^{(2)}(1+R_1)}$ with $R_1 > ||\phi||_{\alpha,\infty}$.

Recall $B_{R_1} = \{Y \in C([0,T], W^{\alpha,\infty}[0,1]) : ||Y||_{\alpha,\infty} < R_1\}$.

Now suppose $Y \in B_{R_1}$. Then $||Y||_{\alpha,\infty} < R_1$ and we get $||F(Y)||_{\alpha,\infty} < R_1$. Thus $F(Y) \in B_{R_1}$ and hence $F(B_{R_1}) \subset B_{R_1}$.

Proposition 1

Proposition

Let $f \in C([0,T], W_0^{\alpha,\infty}[0,1])$ *and* $g \in C([0,T], W^{1-\alpha,\infty,0}[0,1])$. *Then for all* $\xi \in [0,1]$,

$$\left|\int_0^t \int_0^\xi f(s,\gamma) dg_\gamma ds\right| + \int_0^\xi (\xi-\eta)^{-\alpha-1} \left|\int_0^t \int_0^\xi f(s,\gamma) dg_\gamma ds - \int_0^t \int_0^\eta f(s,\gamma) dg_\gamma ds\right| d\eta \leq \Lambda_\alpha(g) b_\alpha^{(3)} \int_0^t \int_0^\xi [(\xi-\gamma)^{-2\alpha} + \gamma^{-\alpha}] \Bigg(|f(s,\gamma)| + \int_0^\gamma \frac{|f(s,\gamma) - f(s,\delta)|}{(\gamma-\delta)^{\alpha+1}} d\delta \Bigg) d\gamma ds. \tag{27}$$

where $b_\alpha^{(3)}$ *is a constant which depends on* α.

Proposition 2

Proposition

Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying the assumptions A3 and A4. Then for all $N > 0$ and $|X_1|, |X_2|, |X_3|, |X_4| \leq N$ for all $X_1, X_2, X_3, X_4 \in \mathbb{R}$,

$$|h(X_1) - h(X_2) - h(X_3) + h(X_4)| \leq M_1|X_1 - X_2 - X_3 + X_4| + M_N|X_1 - X_3|(|X_1 - X_2|^{\zeta} + |X_3 - X_4|^{\zeta}). \quad (28)$$

Lemma 2 (Contraction)

Lemma

Given a constant $R_2 \geq ||\phi||_{\alpha,\infty}$ *there exists* $T_2 > 0$ *and a constant* $0 < C < 1$ *such that,*

$$||F(Y_1) - F(Y_2)||_{\alpha,\infty} \leq C||Y_1 - Y_2||_{\alpha,\infty}$$

for all $Y_1, Y_2 \in B_{R_2} = \{Y \in C([0, T_2], W^{\alpha,\infty}[0,1]) : ||Y||_{\alpha,\infty} \leq R_2\}$.

Theorem (Local Existence - Deterministic Case)

Theorem

Let $0 < \alpha < \frac{1}{2}$, $g \in C([0,T], W^{1-\alpha,\infty,0}[0,1])$. Consider the integro-differential equation

$$Y(t,\xi) = \phi(\xi) + \int_0^t \int_0^\xi A(Y(s,\eta))dg(s,\eta)ds$$

where $t \in [0,T]$, $\xi \in [0,1]$, and A satisfies assumptions A1, A2, A3, and A4. Let $\alpha_0 = \min\{\frac{1}{2}, \zeta\}$. Then the above equation has a unique solution

$$Y \in C([0,T], W^{\alpha,\infty}[0,1]),$$

for $\alpha < \alpha_0$ and $\phi \in W^{\alpha,\infty}([0,1])$ for all for all $T \leq T_0$ where T_0 is a constant.

Theorem 2 (Global Existence - Deterministic Case)

Theorem

Let $0 < \alpha < \frac{1}{2}$, $g \in C([0,T], W^{1-\alpha,\infty,0}[0,1])$. *Consider the integro-differential equation*

$$Y(t,\xi) = \phi(\xi) + \int_0^t \int_0^\xi A(Y(s,\eta))dg(s,\eta)ds$$

where $t \in [0,T]$, $\xi \in [0,1]$, *and A satisfies assumptions A1, A2, A3, and A4. Let* $\alpha_0 = \min\{\frac{1}{2}, \zeta\}$. *Then the above equation has a unique solution*

$$Y \in C([0,T], W^{\alpha,\infty}[0,1]),$$

for $\alpha < \alpha_0$ *and* $\phi \in W^{\alpha,\infty}([0,1])$ *for all* $T > 0$.

Main Theorem

Theorem

Let $\alpha \in (1-H, \alpha_0)$. *If* $\phi \in W^{\alpha,\infty}[0,1]$ *and the function A satisfies the assumptions A1, A2, A3, and A4 then, there exists a unique stochastic process* $Y \in L^0((\Omega, \mathcal{F}, P), C([0,T], W^{\alpha,\infty}[0,1]))$ *which is a solution of the stochastic partial differential equation* (17).

Proof.

The random variable $G = \frac{1}{\Gamma(1-\alpha)} \sup_{0<\eta<\xi<1} |(D_{\xi-}^{1-\alpha} B_{\xi-})(\eta)|$ has moments of all orders. Since the pathwise integral $\int_0^1 A(\eta) dB_\eta$ exists for $1-H < \alpha < \frac{1}{2}$, with $A \in W_0^{\alpha,1}$, we have the estimate $\left| \int_0^1 A(\eta) dB_\eta \right| \leq G ||A||_{\alpha,1}$. Thus the existence and uniqueness follows. □

Main References

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Thank You

Riemann-Liouville Fractional Integrals

Definition

Let $f \in L^1(a,b)$ and $\alpha > 0$. The left-sided and right-sided Riemann-Liouville fractional integrals of f of order α are defined for almost all $x \in (a,b)$ by

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{(1-\alpha)}} dt \tag{29}$$

and

$$I^{\alpha}_{b-} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{(1-\alpha)}} dt \tag{30}$$

respectively, where $(-1)^{\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^{\infty} r^{(\alpha-1)} e^{-r} dr$ is the Gamma function or the Euler integral of the second kind.

Weyl Derivative

Definition

Suppose $I^{\alpha}_{a+}(L^p)$ be the image of $L^p(a,b)$ by the operator I^{α}_{a+} and $I^{\alpha}_{b-}(L^p)$ be the image of $L^p(a,b)$ by the operator I^{α}_{b-}. Let Let $0<\alpha<1$. Then we define the Weyl derivative for almost all $x \in (a,b)$ as;

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha\int_a^x \frac{f(x)-f(t)}{(x-t)^{\alpha+1}}dt\right)1_{(a,b)}(x) \qquad (31)$$

and

$$D^{\alpha}_{b-}f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha\int_x^b \frac{f(x)-f(t)}{(t-x)^{\alpha+1}}dt\right)1_{(a,b)}(x). \qquad (32)$$

The convergence of the integrals at the singularity $t=x$ holds pointwise for almost all $x \in (a,b)$ when $p=1$ and in L^p sense when $1<p<\infty$.