Some Results and Conjectures for Tree Polymers under Weak/Strong Disorder

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3RD FRONTIER PROBABILITY DAYS
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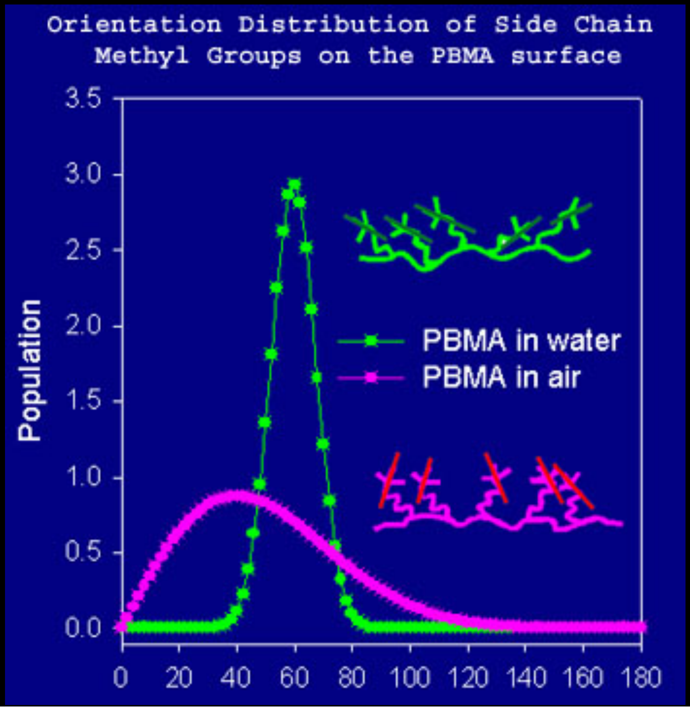


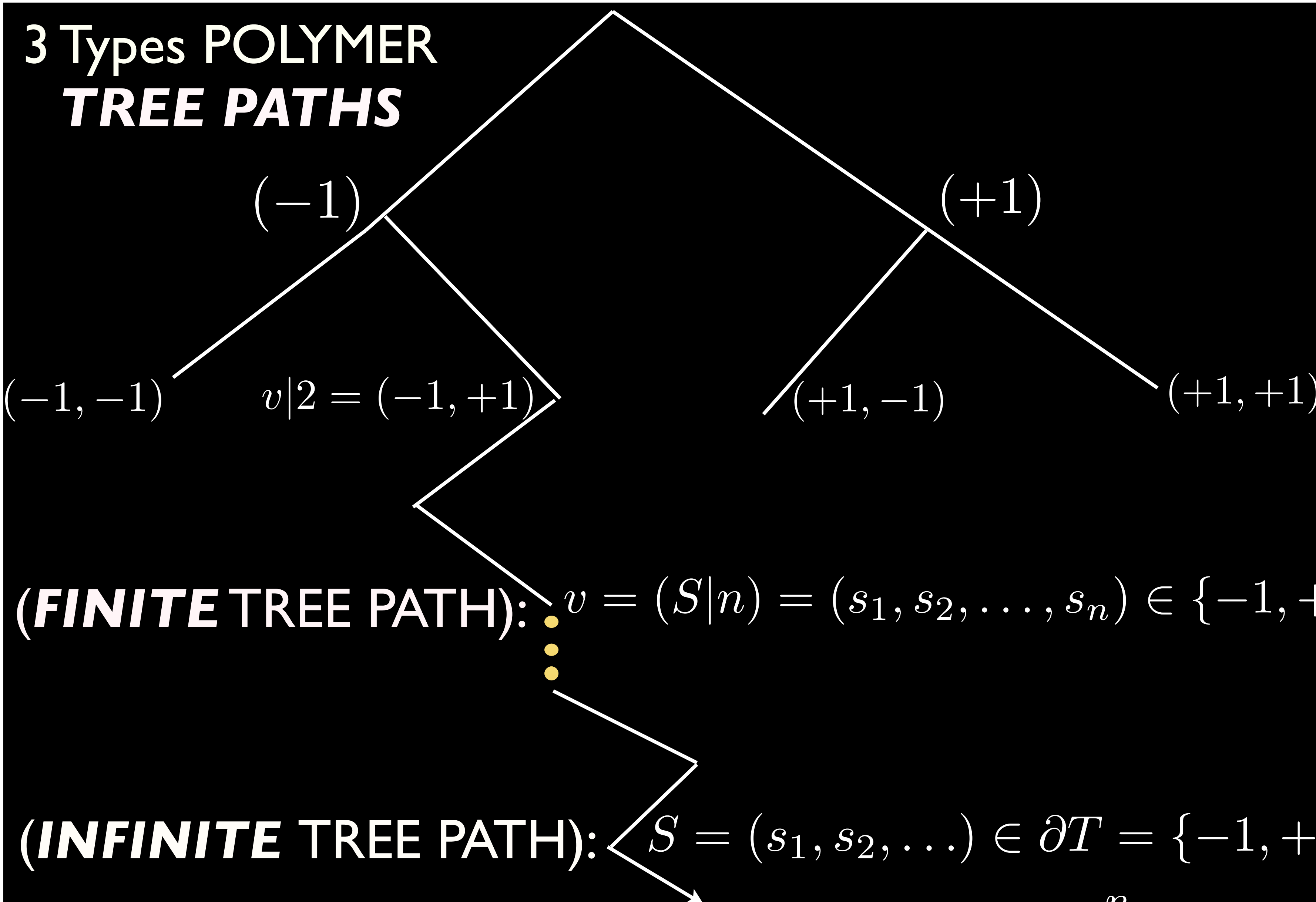
Based on joint work with Stanley C. Williams, Torrey Johnson

[W&W,2010] Tree polymers and multiplicative cascades, in: *Fractals and Related Fields*, ed Julien Barrel, Birkhauser

[J&W,2011] Tree polymers in the infinite volume limit at critical strong disorder, (PREPRINT)

3 Types POLYMER

TREE PATHS



(***FINITE*** TREE PATH): $v=(S|n)=(s_1,s_2,\ldots,s_n)\in\{-1,+1\}^n$

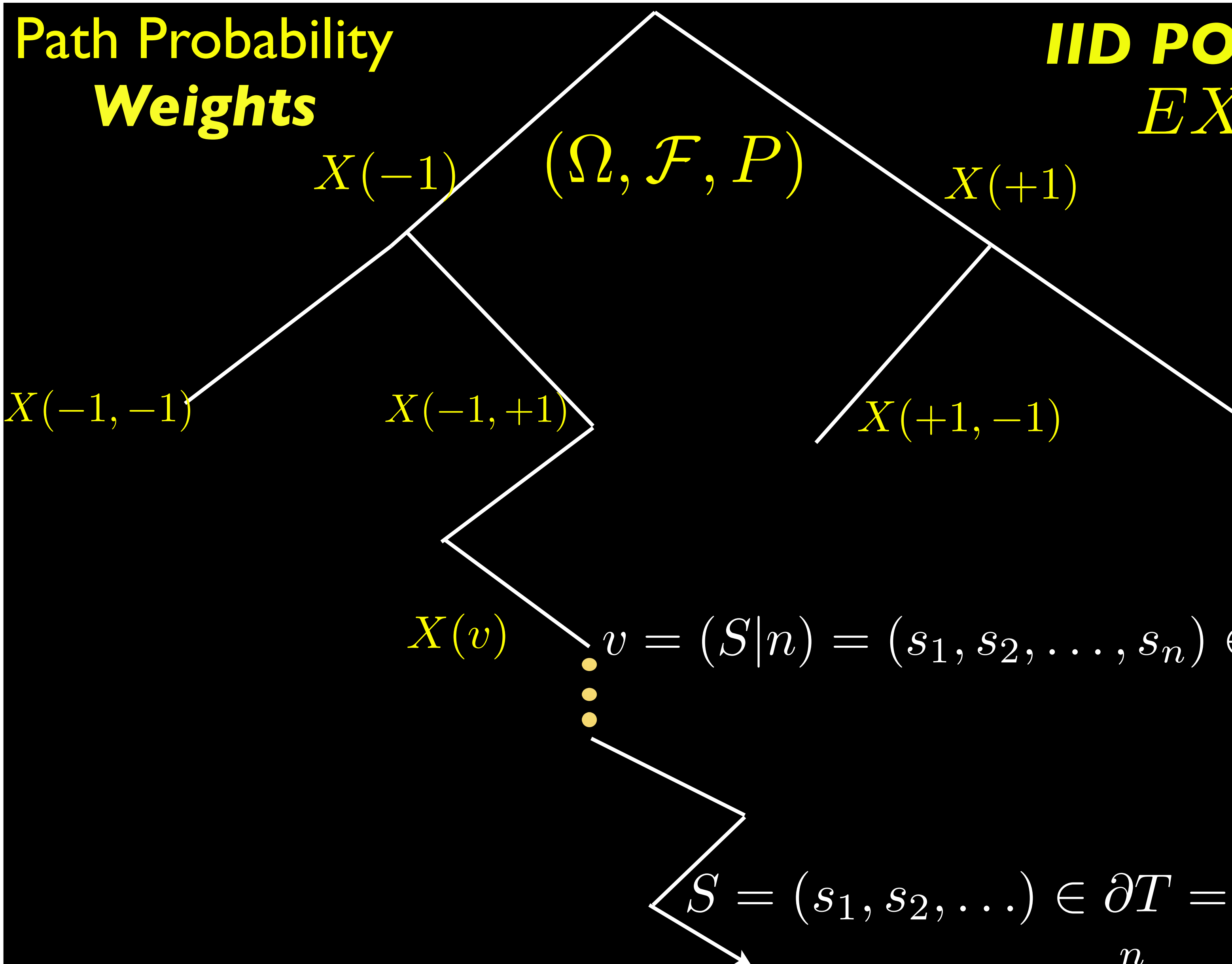
(***INFINITE*** TREE PATH): $S=(s_1,s_2,\ldots)\in\partial T=\{-1,+1\}^\infty$

(***POLYGONAL*** TREE PATH): $n\to(S)_n=\sum_{j=1}^{n}s_j$

Path Probability **Weights**

IID POSITIVE

$EX = 1$



$$v = (S|n) = (s_1, s_2, \ldots, s_n) \in \{-1,+1\}^n$$

$$S = (s_1, s_2, \ldots) \in \partial T = \{-1,+1\}^\infty$$

$$n \longrightarrow (S)_n = \sum_{j=1}^{n} s_j$$

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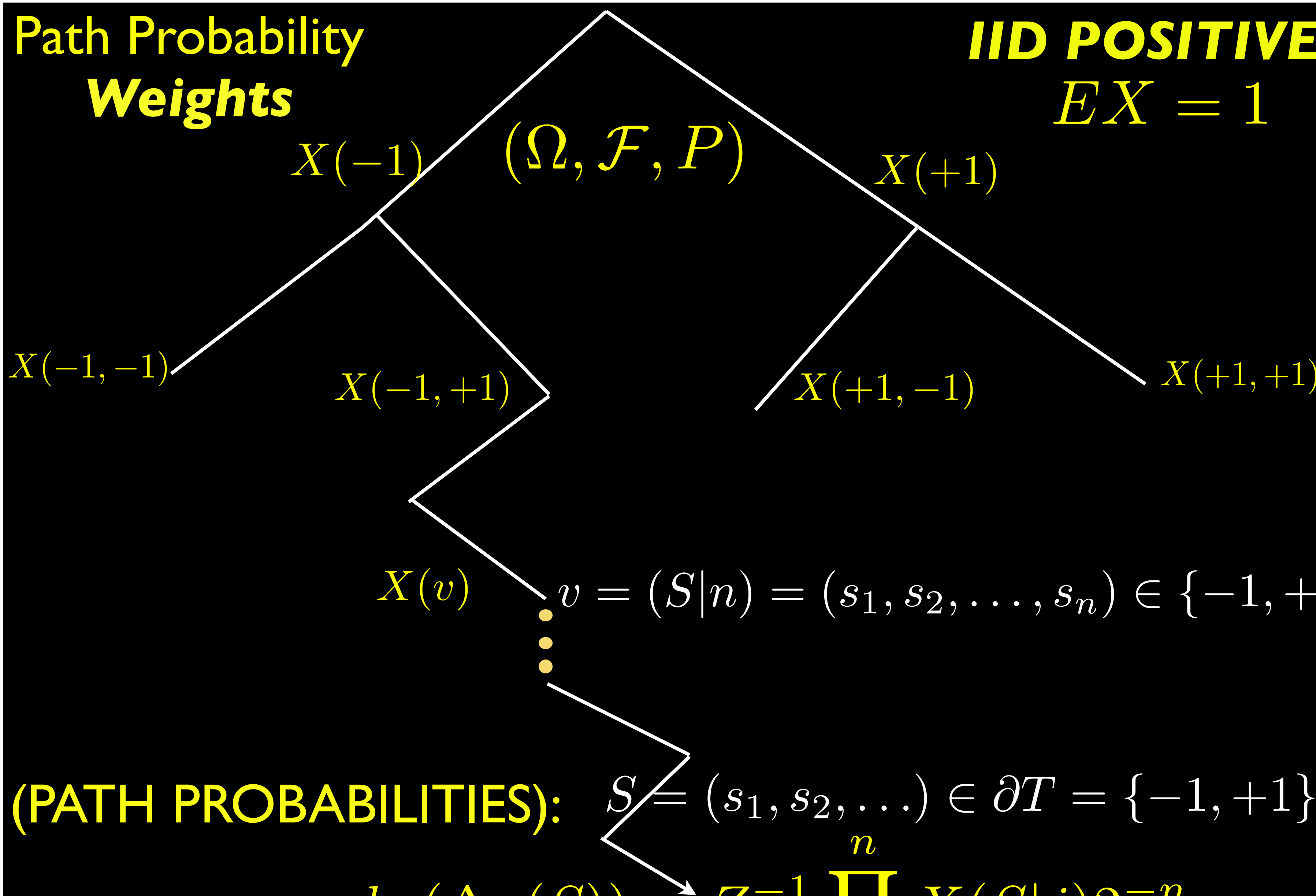


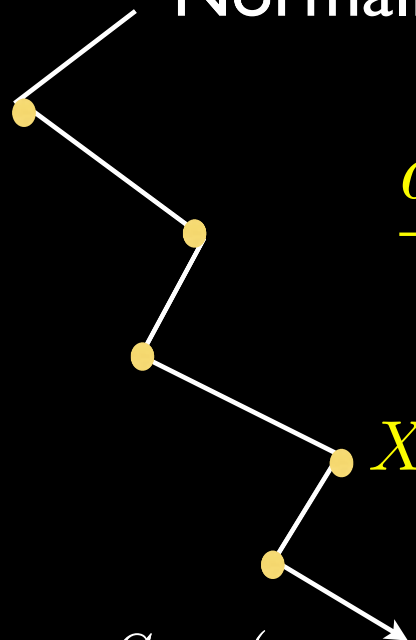
$v = (S|n) = (s_1, s_2, \ldots, s_n) \in \{-1,+1\}^n$

(PATH PROBABILITIES): $S = (s_1, s_2, \ldots) \in \partial T = \{-1,+1\}^\infty$

$$prob_n(\Delta_n(S)) = Z_n^{-1} \prod_{j=1}^{n} X(S|j) 2^{-n}$$

$\Delta_n(S) = \{s \in \partial T : s|n = S|n\}$

Normalized Haar measure: $\lambda(ds)$ on $\{-1,1\}^\infty$



$$\frac{dprob_n}{d\lambda}(S) = Z_n^{-1}\prod_{j=1}^{n} X(S|j)$$

$$S = (s_1, s_2, \ldots) \in \partial T = \{-1,+1\}^\infty \qquad (S)_n = \sum_{j=1}^{n} s_j$$

SIMPLE SYMMETRIC RW $\quad X \equiv 1 \Rightarrow prob_n(ds) = \lambda(ds)$

PROBABILITY LAWS OF $n \to (S)_n$ WELL UNDERSTOOD !

Bolthausen's Disorder Parameterization --

Weak Disorder: $$Z_\infty = \lim_{n\to\infty} Z_n > 0 \ a.s.$$

Strong Disorder: $$Z_\infty = \lim_{n\to\infty} Z_n = 0 \ a.s.$$

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Theorem (Kahane-Peyriere `76, Biggins `76, W., Williams, `94)

Weak Disorder if and only if $EX \ln X < \ln 2$

Strong Disorder if and only if $EX \ln X \geq \ln 2$

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THEOREM (Kahane-Peyriere 1976 ``Live/Die" Criteria)

Weak Disorder if and only if $EX \ln X < \ln 2$

Strong Disorder if and only if $EX \ln X \geq \ln 2$

(Disorder Strength vs Branching Rate)

TYPICAL POLYMER PROBLEMS FOR:

$$(S)_n = \sum_{j=1}^{n} s_j$$

A. a.s.Non-ballistic ? (a.s. LLN---Weak/Strong ?)

B. Infinite Volume Limit ? $prob_\infty(ds) = \lim_{n\to\infty} prob_n(ds)\ a.s.?$

C. a.s. Diffusive (sub/super) ? (a.s.CLT---Universal/Variance ?)

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$$prob_n(\{s \in \partial T : \frac{(s)_n}{\sqrt{n}} \leq x\}) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz\ a.s.\ \ ?$$

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SIMPLE SYMMETRIC RW BENCHMARK: $prob_n(ds) = \lambda(ds)$

ANSWER A: Always non-ballistic (in a weak sense) regardless of disorder type.

$$E_{prob_n}|\frac{(S)_n}{n}| \to 0 \quad a.s. \qquad \text{as n} \to \infty$$

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NOTE: a..s. Strong Law requires answer to Question B
Existence of $prob_\infty$

e.g. $prob_\infty(s \in \partial T : \frac{(s)_n}{n} \to 0) = 1 \quad \text{a.s.}$

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e.g., $$prob_\infty(s \in \partial T : \frac{(s)_n}{n} \rightarrow 0) = 1 \quad \text{a.s.}$$

Similarly, law of iterated logarithm, and other a.s. , a.s. laws

ANSWER B. SPECIAL CASES

PROPOSITION B2 (Johnson, W.) In the case of weak disorder

a. $$prob_\infty(ds) = \lim_{n\to\infty} prob_n(ds)$$ exists a..s.

In the case of critical strong disorder, for each finite $F \subset N$

b. $$\widehat{prob}_\infty(F) = \lim_{n\to\infty} \widehat{prob}_n(F) \quad \text{in probability}$$

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PROOF IDEA:

WD: Martingale Convergence Theorem

$$M_n(f) = Z_n \int_{\partial T} f(s) prob_n(ds), \quad n \geq 1.$$

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SD: Seneta-Heyde scaling & Derivative Martingale for BRW.
Aidekon and Shi (2011), Biggins and Kyprianou (2004)

If μ is a probability on group ∂T , $\hat{\mu}(F) = \int_{\partial T} \prod_{j\in F}(s_j)\mu(ds)$ is Fourier transform.

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Conventional Wisdom: General Nonexistence for SD.

Remark: Size-Biased Existence for SD (W. & Williams 1996)

Long-chain Diffusivity

THEOREM (Bolthausen, 1989)

Assume lognormally distributed weights: $X = e^{\beta W}$

Then for $\beta < \sqrt{2} < \beta_c = \sqrt{2 \ln 2}$ one has the diffusive scaling

$$\int_{\partial T} \left(\frac{(S)_n}{\sqrt{n}} \right)^2 prob_n(dS) \to 1 \quad a.s.$$

Also a.s.asymptotically normal distribution

$$M_n(\frac{r}{\sqrt{n}}) \equiv \int_{\partial T} e^{\frac{(S)_n}{\sqrt{n}} r} prob_n(dS) \to e^{\frac{r^2}{2}} \qquad -\delta < r < \delta$$

NOTE: The full weak disorder regime is $\beta < \beta_c = \sqrt{2 \ln 2}$

i.e., since weak disorder is:

$$\frac{\beta^2}{2} = E \frac{X}{EX} \ln \frac{X}{EX} < \ln 2$$

PROPOSITION C1 (Weak Disorder)W.&Williams (2010)

Under weak disorder the following limit exists a.s. :

$$\lim_{n\to\infty} \frac{\ln M_n(r)}{n} = \ln\cosh(r) \quad a.s.$$

THEOREM C1. (W&W 2010): Assume $EX^{1+\epsilon} < \infty$

For the tree polymer model weak disorder implies diffusive scaling and asymptotic normality almost surely.

That is, a.s.

$$M_n(\frac{r}{\sqrt{n}}) \equiv \int_{\partial T} e^{\frac{(S)_n}{\sqrt{n}}r} prob_n(dS) \to e^{\frac{r^2}{2}} \quad -\delta < r < \delta$$

REMARK: Comets and Yoshida (2006), AoP, Proved CLT for d+1-dimensional lattice polymers, $d \geq 3$. Cov $\Sigma = \frac{1}{d} I_d$.

Proposition C3. (Strong Disorder) W.&Williams (2010)

Under strong disorder the following limit exists a.s. :

$$F(r) = \lim_{n\to\infty} \frac{\ln M_n(r)}{n}$$

$$= \ln\cosh(r) + \frac{\ln EX^{h(r)} + \ln\left(p_r^{h(r)}(+) + p_r^{h(r)}(-)\right)}{h(r)}$$

where $h = h(r)$ solves

$$E\left\{\frac{X^h}{EX^h}\ln\frac{X^h}{EX^h}\right\} = \epsilon\left(\overline{p}_r^h(+), \overline{p}_r^h(-)\right)$$

$$\epsilon(p,q) = -p\ln p - q\ln q \ ; \quad \overline{p}_r^h(\pm) = \frac{p_r^h(\pm)}{p_r^h(+) + p_r^h(-)}$$

SPECIAL CASE: $X = e^{\beta W - \frac{\beta^2}{2}}$

$$F(r) = r\tanh(rh(r)) + \beta^2 h(r) - \beta_c \beta$$

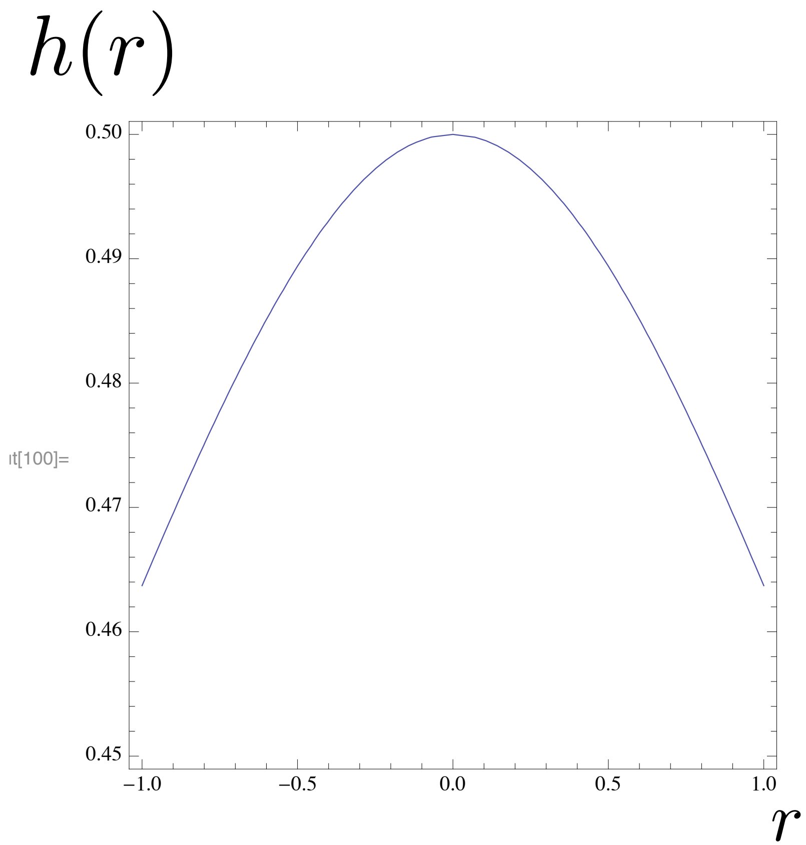
where

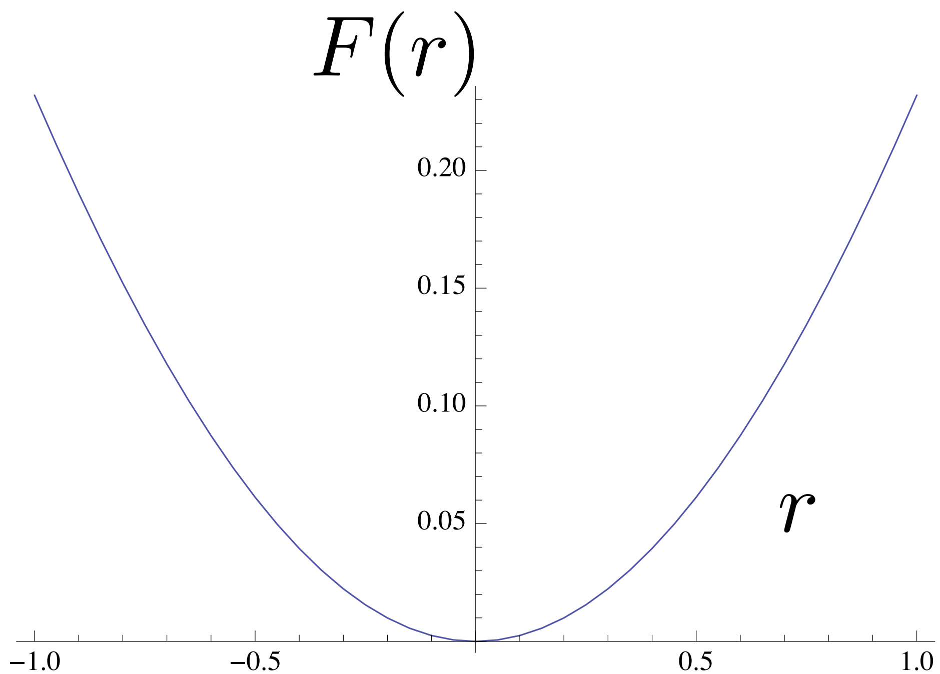
$$\beta^2 h^2(r) + 2rh(r)\tanh(rh(r)) - 2\ln\cosh(rh(r)) = \beta_c^2$$

$$F(r) = r\tanh(rh(r)) + \beta^2 h(r) - \beta\beta_c$$

$$\beta^2 h^2(r) + 2rh(r)\tanh(rh(r)) - 2\ln\cosh(rh(r)) = \beta_c^2$$

$$\beta = 2\sqrt{2\ln 2} \quad > \beta_c$$





CONJECTURE (J. , W. 2011) Diffusive scaling and variance $\sigma^2 = \sigma^2(\beta)$ under strong disorder

where

$$\sigma^2(\beta) = \frac{2\beta\beta_c - \beta_c^2}{\beta^2}, \quad \beta \geq \beta_c$$

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--SKETCHES OF PROOFS--

SIZE BIAS TOOL: (W., Williams 1994)

Define the ***size-bias*** probability on $\Omega \times \partial T$

$$Q(d\omega \times ds) = P_s(d\omega)\lambda(ds)$$

where on $\sigma(X_v : |v| \leq n)$

$$P_s(d\omega) = \prod_{j=1}^{n} X_{s|j}(\omega) P(d\omega)$$

SIZE BIAS TOOL: (W., Williams 1996 TAMS)

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$$P_\omega(ds) = prob_n(\omega, ds)$$

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$$\mathcal{Q}_\Omega(d\omega) \equiv \mathcal{Q} \circ \pi_\Omega^{-1}(d\omega) = Z_n(\omega) P(d\omega)$$

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$$\mathcal{Q}_\Omega(d\omega) \equiv \mathcal{Q} \circ \pi_\Omega^{-1}(d\omega) = Z_n(\omega)P(d\omega)$$

$$\mathcal{Q}_{\partial T}(ds) \equiv \mathcal{Q} \circ \pi_{\partial T}^{-1}(ds) = \lambda(ds)$$

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$$P_s(d\omega) = \prod_{j=1}^{n} X_{s|j}(\omega) P(d\omega) \quad \text{on} \quad \sigma(X_v : |v| \leq n)$$

LEMMA (W,W, `94): On $\mathcal{F} = \sigma(X_v : v \in T)$

Weak disorder : $\mathcal{Q}_\Omega(d\omega) << P(d\omega)$

Strong disorder: $\mathcal{Q}_\Omega(d\omega) \perp P(d\omega)$

PROOF IDEAS:

WD: $$\frac{dQ_\Omega}{dP}(\omega) = Z_\infty(\omega) < \infty$$

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SD:

$$Z_n \geq \prod_{j=1}^{n} X_{s|j} 2^{-n} = \exp\{n(\frac{1}{n}\sum_{j=1}^{n} X_{s|j} - \ln 2)\}$$

$$\sim \exp\{n(EX \ln X - \ln 2)\}$$

$$\rightarrow \infty \quad \text{if} \quad \mathrm{EX} \ln \mathrm{X} > \ln 2$$

But, positive martingale property implies

$$P(Z_\infty < \infty) = 1$$

i.e. $Q_\Omega(d\omega) \perp P(d\omega)$

Calculate

$$F(r) = \lim_{n\to\infty} \frac{\ln M_n(r)}{n}$$

WD:

$$\frac{Z_n M_n(r)}{\cosh^n(r)} \qquad -\delta < r < \delta$$

Kahane's T-Martingale

SD:

Via SIZE-BIASED Borel-Cantelli limsup and liminf calculations

OR

Via a vector cascade T-martingale coding (Displ., Weight)

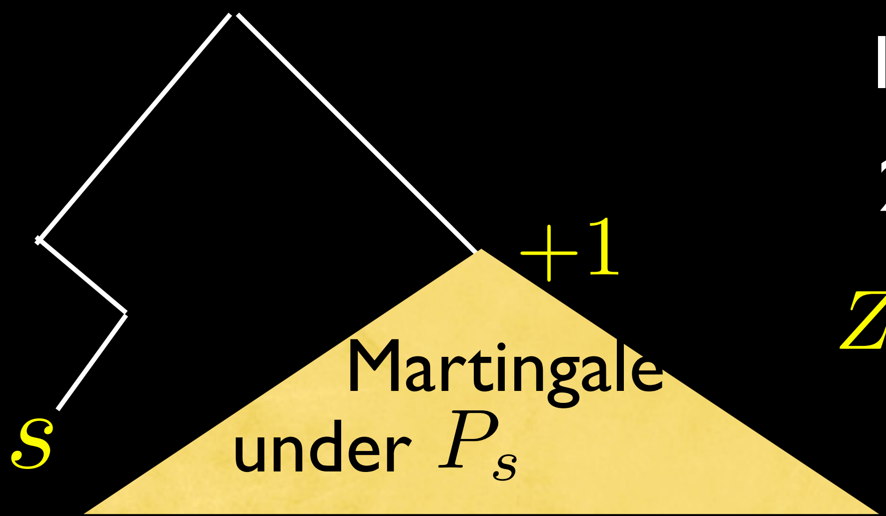
PROPOSITION B2 (W., Williams 1996 TAMS)

In the case of strong disorder there is a random path $\tau = \tau(\omega)$ such that

$$prob_\infty(ds) = \delta_\tau(ds) \quad Q_\Omega - a.s.$$

PROOF IDEA:

Strong disorder $Q_\Omega - a.s.$ makes $Z_\infty = \infty$



$\Rightarrow Z^+_\infty(\omega) < \infty$

1. Fix a path s say $s_1 = -1$

2. So now fix $\omega \in [Z_\infty = \infty]$

$Z^+_\infty(\omega) < \infty \Rightarrow \tau_1(\omega) = -1$

Theorem C1 (Proof Ideas)

For the tree polymer model weak disorder implies diffusive scaling and asymptotic normality almost surely.

That is

$$\lim_{n\to\infty} M_n\left(\frac{r}{\sqrt{n}}\right) = e^{\frac{r^2}{2}} \quad |r| < \delta \ \text{ a.s.}$$

Sketch of Proof:

$$EZ_n M_n(r) = \cosh^n(r)$$

$$\frac{Z_n M_n(r)}{\cosh^n(r)} \quad -\delta < r < \delta$$ Kahane's T-MARTINGALE

$$\frac{d}{dr}\frac{Z_n M_n(r)}{\cosh^n(r)} \quad -\delta < r < \delta$$ (Signed) T-MARTINGALE

$$m_n'(r) = \frac{d}{dr}\frac{Z_n M_n(r)}{\cosh^n(r)} = \sum_{j=1}^{n} m_{n,j}(r)$$

$$E|m_n'(r)|^q \le C \sum_{j=1}^{\infty} \left(\frac{EX^q}{2^{q-1}} \right)^j$$

WEAK DISORDER $\Longrightarrow$

$$\exists \quad 1 < q < 2 \quad \text{such that} \quad \frac{\mathrm{EX^q}}{2^{\mathrm{q}-1}} < 1$$

$$\Longrightarrow \qquad \frac{Z_n M_n(r)}{\cosh^n(r)} \to Y(r) \quad |r| < \delta$$

$$M_n\left(\frac{r}{\sqrt{n}}\right) = Z_n^{-1} \frac{Z_n M_n\left(\frac{r}{\sqrt{n}}\right)}{\cosh^n\left(\frac{r}{\sqrt{n}}\right)} \cosh^n\left(\frac{r}{\sqrt{n}}\right) \to \overset{1}{\cancel{Z_\infty^{-1}}} Y(0) e^{\frac{r^2}{2}} = e^{\frac{r^2}{2}}$$

$\blacksquare$

THANK YOU