The rate of escape for random walks on solvable groups

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- G can be viewed as a (length) metric space $(G, |\cdot|_G)$, which is the Cayley graph of G with respect to S.
- Changing generating sets gives equivalent metrics on G and quasi-isometric Cayley graphs.

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A random walk, X_n, on G whose increments are distributed according to such a measure is said to be simple and symmetric. In particular, $X_n = ex_1x_2 \cdots x_n$ where x_i is the ith increment of the random walk.

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Big question: On a given group does the escape exponent of a simple symmetric random walk depend on the choice of finite generating set?

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- Non-amenable groups: $\alpha = 1$ for all simple symmetric random walks (classical).
- Outside of these classes the escape exponent is known only on particular groups and only for particular families of measures (Revelle, Erschler).

Let H and K be groups. The semidirect product $G = H \rtimes_\phi K$, $\phi : K \to Aut(H)$ is given by the following multiplication rule:

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Suppose the random walk, $X_n = (W_n, Y_n)$, on G has increments $x_i = (w_i, y_i)$. Then

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Observe that $|Y_n|_G = |Y_n|_K$, but the story is more complicated for W_n.

A finitely generated subgroup H of G is said to be *strictly exponentially distorted* if there exists a $c > 0$ such that

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Examples: Sol; $BS(1,n)$, $n \ = 1$.

Theorem

Let $G = H \rtimes_\phi \mathbb{Z}^d$. If H is finitely generated and strictly exponentially distorted in G, then for any simple symmetric random walk $X_n = (W_n, Y_n)$ on G there exists $C > 0$ such that

$$\mathbb{E}_{\mu(n)}|Y_n|_G \le \mathbb{E}_{\mu(n)}|X_n|_G \le C(\mathbb{E}_{\mu(n)} \max_{i<n} |Y_n|_G + \mathbb{E}_{\mu(n)}|Y_n|_G + \log n)$$

for all $n \ge 1$.

Strictly exponential distortion and the rate of escape

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Example: $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$, $|tr(\phi)| > 2$.

Theorem

Let $G = H \rtimes_\phi \mathbb{Z}^d$ be finitely generated. Fix a generating set S of G. If $\pi_H(S)$ generates a strictly exponentially distorted subgroup of G, then for any symmetric random walk supported on S there exists $C > 0$ such that

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Another version of the theorem

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Idea of the proofs.

Show that for all $z \in \mathbb{Z}^d, h \in H$, there exists $q \geq 1$ such that $|\phi^z \cdot h|_H \leq q^{|z|}|h|_H$. Then use strictly exponential distortion to bring this estimate into $|\cdot|_G$. □