Recurrence of Tandem Queues under Random Perturbations

Florian Sobieczky

13. März 2011

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M/M/1-Queues

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2. **Serving Times** S_j between sales ($j \in \mathbb{N}$)
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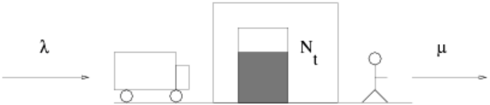

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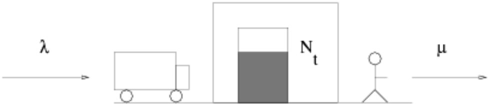

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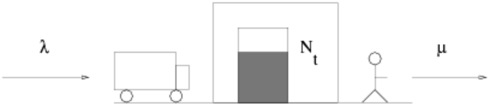

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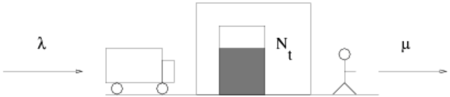

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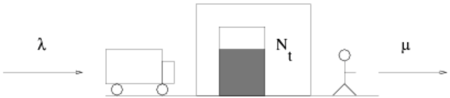

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Buying-rate λ Inventory-Size N_t Inquiry-Rate μ

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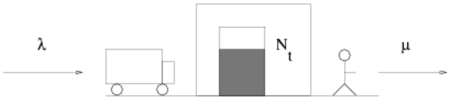

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- Expected asymptotic size of inventory: (by ergodicity) $\lim_{t \to \infty} \mathbb{E}[N_t] = \mathbb{E}_\pi[N_t] = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$

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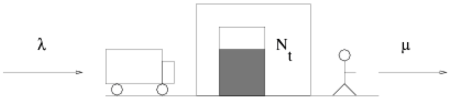

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Quarter-plane Random Walk

1. **Tandem** M/M/1 Queueing network: $((\lambda_1, \mu_1), (\lambda_2, \mu_2))$:

   Stock-Sizes M_t, Shop-Inventory N_t

2. **2 Interarrival Times** T_j^1, T_j^2, $(j \in \mathbb{N})$

   $T_j^1 \sim \mathrm{Exp}(\lambda_1)$, $T_j^2 \sim \mathrm{Exp}(\lambda_2)$, i.i.d.

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   'Every item leaving Stock enters Shop' $\Leftrightarrow$ $\mu_1 > \lambda_1 = \lambda_2 =: \lambda$

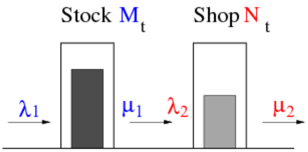

Figure 4: A series of two M/M/1-queues applied to inventory-model with stock. The product form of the invariant distribution of this simple queueing network follows from Burke's theorem: P.Burke: 'The output of a Queueing System', Operations Research, Vol. 4, No. 6 (1956), p. 699-704

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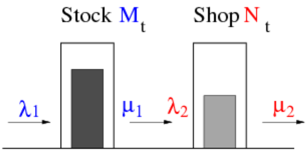

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Burke's Theorem I

- Stability :⇔ both queues (stock and shop) remain finite

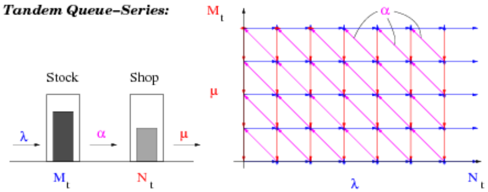

Figure 5: a.) In Equilibrium: $\pi_{k,l} := \mathbb{P}[M_t = k, N_t = l] = (1-\rho_2)\rho_2^k(1-\rho_2)\rho_2^l$ where $\rho_1 = \frac{\lambda}{\alpha}$, $\rho_2 = \frac{\lambda}{\mu}$. (Product Measure) b.) (Imbedded random walk on quarter plane $\mathbb{Z}^2$)

Buying rate $\lambda = \lambda_1 = \lambda_2$; Transport rate $\alpha = \mu_1$; Inquiry rate $\mu = \mu_2$

- **Theorem:** Stability ⇔ $\lambda \leq \alpha$ and $\lambda \leq \mu$

Proof: Imbedded Markov chain is space-homogeneous Random Walk on $\mathbb{Z}_+^2$.
Recurrence is guaranteed by Theorem 1.2.1(ii) in: G. Fayolle, R. Iasnogorodski, V. Malyshev: '*Random Walks in the Quarter Plane*', Springer 1999
Positive recurrence follows from geometric decay of π_1 and π_2

Burke's Theorem II

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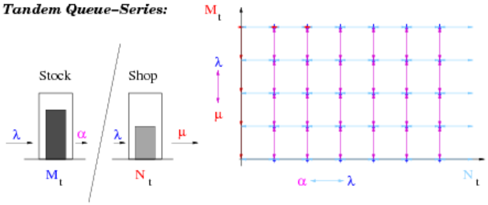

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Random Weights

- Transition rates λ_k and μ_k dependent on queue-size $N_t = k$

$\mathbb{P}[N_{n+1} = k+1 \mid N_n = k] = \frac{\lambda_k}{\lambda_k+\mu_k}$; $\mathbb{P}[N_{n+1} = k-1 \mid N_n = k] = \frac{\mu_k}{\lambda_k+\mu_k}$

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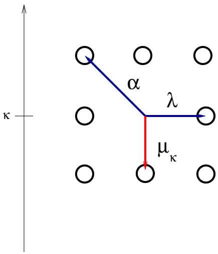
Example: Random vertikal conductances

$\mu_k \sim$Binom, i.i.d: $\mathbb{P}[\mu_k = \lambda + \epsilon] = p = 1 - \mathbb{P}[\mu_k = \lambda - \epsilon]$



The Process is positive recurrent iff $p > \frac{1}{2}\left(1 + \frac{\epsilon}{\lambda}\right)$.

Other ‘Perturbed’ Tandem-Queues

A continuous-time random walk (such as a queueing-system) consists of three steps: Let $j = 0$

1. Wait exp.-distrib. Time with rate $\lambda + \mu$
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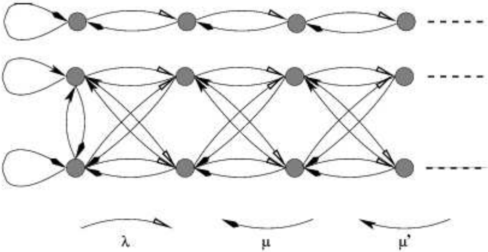

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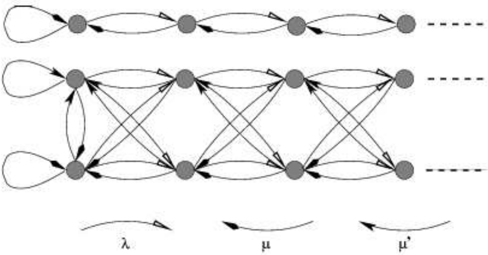

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Two different modifications of step 2.: Perturbed chain is projection of chain on covering graph onto ‘original’ graph (integer-line); Projections must be Markov-Processes!

2’. First pick layer (where to end up) with probability 1/2, then use probabilities depending on layer where currently situated:

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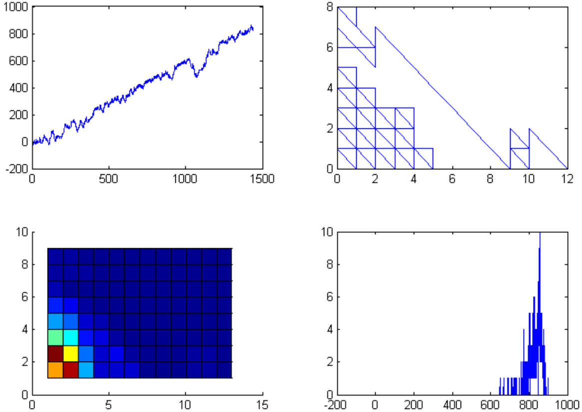

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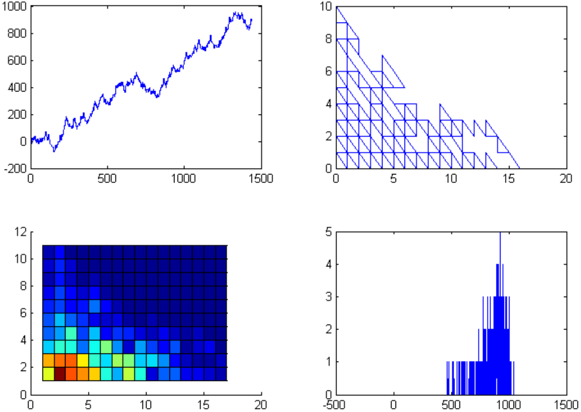

Literatur

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- G Fayolle, V. A. Malyshev, M. V. Men'shikov: Topics in the Constructive Theory of Countable Markov Chains, 1995
- F. Sobieczky, G. Rappitsch, E. Stadlober: Tandem Queues for Inventory Management under Random Perturbations, 2010

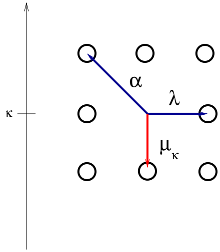

Abbildung: Thank you for your interest!