Phase transition and reconstruction for the Glauber dynamics on trees

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Based in joint work with Daniel Stefankovic, Juan Vera, Eric Vigoda and Linji Yang

Spin systems

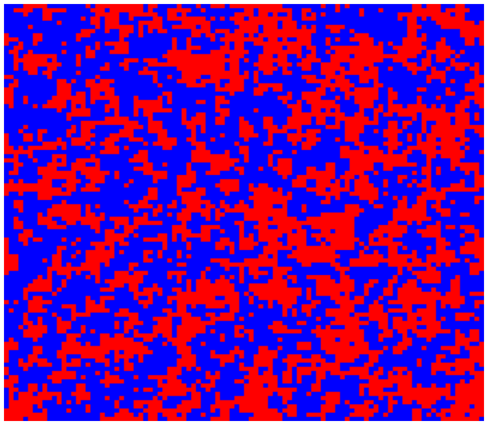
- Given an underlying graph G, a configuration is an assignment of a given set of spins to the vertices of G.

- A given configuration has a weight given by

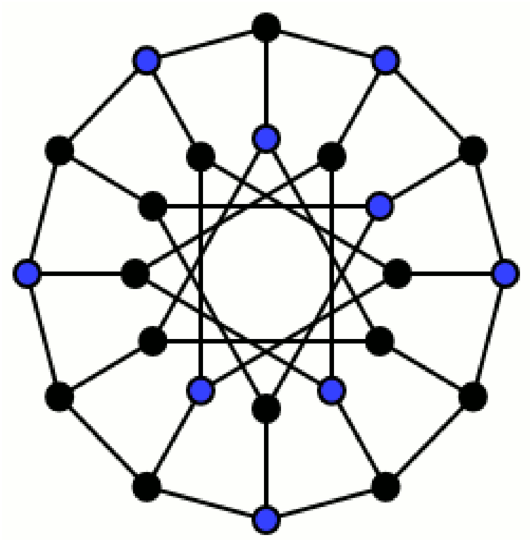
$$\mu(\sigma) = \prod_{v \in G} \psi(\sigma_v) \prod_{v \sim w} \varphi(\sigma_v, \sigma_w)$$

Spin systems

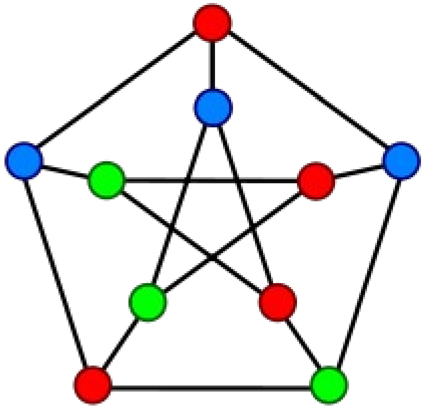
- Ising model
- Hard-core model
- Proper coloring



$$\psi(x) = e^{lx}$$
$$\varphi(x,y) = e^{\beta xy}$$



$$\psi(x) = \lambda^{x}$$
$$\varphi(x,y) = (1 - xy)$$



$$\psi(x) = 1$$
$$\varphi(x,y) = (1 - \delta_{x,y})$$

Glauber dynamics

- Markov Chain Monte Carlo Algorithm for sampling.
- Visit a vertex unif. at random, update the vertex according to its conditional marginal.
- How fast it converges to stationarity?
- Inverse gap, mixing time?

Gibbs measure
Uniqueness - Extremality

- Gibbs measure: Consistent extension to an infinite graph.

- Uniqueness?

- Extremality? (Reconstruction).

Hard-core model

- For regular tree of branching b, let $\omega\left(1+\omega\right)^{d}=\lambda,$

$$\omega_u = \frac{1}{d-1} \qquad \omega_r \approx \frac{\log d}{d}$$

- Uniqueness → Threshold for hardness in general graphs [Sly, 2011].
- Reconstruction → ?
- What happens for trees?

Glauber for hard-core on trees

[Martinelli, Sinclair, Weitz]:

- Fast mixing of glauber for free boundary, any fugacity.
- Fast mixing for any boundary, but $\omega < \frac{\omega_r}{2}$

[R., Stefankovic, Vera, Vigoda, Yang]:

- Phase transition at the reconstruction threshold!

Phase Transition

- For $\omega = (1-\delta)\ln b/b$ it is the case that

$$\Omega(n) \leq T_{\text{relax}} \leq O(n^{1+o_b(1)}).$$

- For $\omega = (1+\delta)\ln b/b$: it is the case that

$$T_{\text{relax}} \leq O(n^{1+\delta+o_b(1)})$$

And also, there exists a seq. of boundaries such that

$$T_{\text{relax}} = \Omega(n^{1+\delta/2-o_b(1)}).$$

Upper bound ?

- Algorithmic flavor...
- Block dynamics.
- Mixing in the star graph?
- Coupling argument.

Lower bound?

• What is the role of the nonextremality of the broadcasting measure?
• Existence of a 'reconstruction squeme' A.

• Conductance bound:

$$\text{Relaxation Time} = \Omega\left(\frac{\text{effectiveness}(A)}{\text{sensitivity}(A)}\right)$$

Effectiveness - Sensitivity

- Effectiveness: Correlation between the output of A and the actual spin.

- Sensitivity: $E_\sigma \nabla A(\sigma_h)$

$$\nabla f(x_1, \ldots, x_k) = \frac{1}{k} \sum_i |f_i(x_1, \ldots, x_k) - f(x_1, \ldots, x_k)|$$

Parsimonious algorithm

- If all the children are unocupied the guess is that the parent is occupied.
- If some child is occupied the guess is that the parent is unocupied.

- Sensitivity is tractable!

Thank you!