Universality for Wigner Matrices

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Wigner matrices

We consider $N \times N$ hermitian matrices with entries $(h_{\ell k})$

$$h_{\ell k} = \begin{cases} N^{-1/2}(x_{\ell k} + i y_{\ell k}) & \text{si } \ell < k \\ N^{-1/2}\sqrt{2} x_{\ell \ell} & \text{si } \ell = k \\ \bar{h}_{k \ell} & \text{si } \ell > k \end{cases}$$

where $x_{\ell k}$ y $y_{\ell k}$ are all independent and have mean zero and variance $\frac{1}{2}$.

Semicircle

Let $F_N(x)$ be the empirical distribution of the eigenvalues

$$F_N(x) = \frac{1}{N}\#\{\mu \leq x\}$$

then

$$F_N(x) \to F_{sc}(x) = \int_{-\infty}^{x} \rho(t)\, dt$$

where

$$\rho(t) = \frac{1}{2\pi}\sqrt{4 - t^2}\, 1_{|t|\leq 2}$$

(Wigner)

Correlation functions

Let $f(x_1, ..., x_N)$ be the probability density for the eigenvalues disregarding order. The semicircle law is the limiting marginal for any x_i. Define the m-point correlation function as:

$$R_m(x_1, ..., x_m) = \frac{N!}{(N-m)!} \int_{R^{N-m}} f(x)\, dx_{m+1} \dots dx_N,$$

We want to find limits for these functions.

GUE and Dyson's sine kernel

The special case where the variables $x_{\ell k}$ and $y_{\ell k}$ are gaussian has been studied thoroughly. The joint density is known explicitely

$$P(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{Z_n} e^{-N \sum_{k=1}^{n} \lambda_k^2} \prod_{j<k} |\lambda_j - \lambda_k|^2$$

$$= \frac{1}{Z_n} e^{-N \sum_{k=1}^{n} \lambda_k^2} \Delta(\lambda)^2$$

GUE and Dyson's sine kernel

In particular, it is known that

$$\lim_{N\to\infty} \frac{1}{(N\rho(u))^2} R_2^N \left(u + \frac{t_1}{N\rho(u)}, u + \frac{t_2}{N\rho(u)} \right)$$

$$= \det \left(\frac{\sin \pi(ti - tj)}{\pi(ti - tj)} \right)_{i,j=1,2}$$

Moreover, all correlation functions ($m \geq 2$) have limits that can be expressed in terms of this kernel.

GUE and Dyson's sine kernel

The original proof of this result used explicit formulas for R_2 in terms of the (normalized) Hermite orthogonal polynomials h_N. This reduces it to Plancherel-Rotach asymptotics for these polynomials and the formula

$$R_2^N(x_1, x_2) = \det(K_N(x_1, x_2))_{i,j=1,2}$$

where

$$K_N(x_1, x_2) = e^{-\frac{N}{2}(x_1^2+x_2^2)} \frac{h_{N-1}(x_1)h_N(x_2) - h_{N-1}(x_2)h_N(x_1)}{x_1 - x_2}$$

Universality

It had been conjectured for a long time that the sine kernel result was true for any Wigner matrix.
Another instance of the universality phenomenon was proved in 1999 by Soshnikov. Let Λ_1 be the largest eigenvalue of H, he proved that

$$P[\Lambda_1 > 2 + sN^{-2/3}] \to \exp\left(-\int_0^\infty (x-s)^2 q(s)\, ds\right)$$

where q is the solution to Painleve II equation $q'' = xq + 2q^3$ with asymptotics $q(x) \sim Ai(x)$ when $x \to +\infty$. This is the Tracy-Widom law.

Ornstein-Uhlenbeck

It is possible to define a diffusion process on hermitian matrices whose invariant measure is GUE. It acts independently on each $x_{\ell k}$ and $y_{\ell k}$ and its generator is

$$L = \frac{1}{4}\frac{\partial^2}{\partial x^2} - \frac{x}{2}\frac{\partial}{\partial x}$$

If started from any matrix H, the distribution of the process after time t coincides with that of

$$e^{-t/2}H + (1 - e^{-t})^{1/2}V$$

where V is GUE.

Ornstein-Uhlenbeck

The OU dynamics on the entries translates into the eigenvalues. They perform independent OU processes conditioned on not hitting each other. The SDE is

$$d\lambda_i = \frac{1}{\sqrt{2}} db_i - N\frac{\lambda_i}{2} dt + \sum_{i \neq j} \frac{2}{\lambda_i - \lambda_j} dt$$

Johansson's work

Let H be a Wigner matrix and V be GUE. Johansson made use of the OU process to prove that matrices of the form

$$H + aV$$

(a fixed) also satisfied that their correlation functions converged to the ones defined through the sine kernel.

Johansson's formula

The proof by Johansson was based on an explicit formula for the transition probabilities of the OU process on the eigenvalues. These rely on the Harish-Chandra/Itzykson-Zuber formula

$$\int_{U(N)} e^{-\frac{N}{2}a^2 \mathrm{Tr}(U^{-1}RU-H)^2}\, dU$$

$$= \frac{1}{\Delta_N(x)\Delta_N(y)} \det\left(e^{-N(x_j-y_k)^2/(2a^2)}\right)_{j,k=1}^{N}$$

where x are the eigenvalues of R and y those of H.

Steepest descent

With that, the measure on the perturbed ensemble looks like

$$\left(\frac{N}{2\pi a^2}\right)^{N/2} \frac{\Delta_N(x)}{\Delta_N(y)} \det\left(e^{-\frac{N}{2a^2}(x_j-y_k)^2}\right)_{j,k=1}^{N} dP^{(N)}(H)$$

where $P^N(H)$ is the measure on the original Wigner matrices (y are the eigenvalues of H).
The previous formula is reexpressed, through Fourier transform and Cauchy's theorem, in terms of a line integral on the complex plane. This was analyzed through a stationary phase procedure.

Steepest descent

This procedure relies heavily on a strong form of the semicircle limit that bounds the Stieltjes transform

$$\left| \frac{1}{N} \sum_{i=1}^{N} \log(z - y_j) - \frac{2}{\pi} \int_{-1}^{1} \log(z - y) \sqrt{1 - y^2} \, dy \right| \leq \frac{C}{N^\eta}$$

($\eta < 1/2$) when z is uniformly away from the real line.

Local semicircle

Under some decay conditions Erdös, Schlein and Yau have proved an even stronger local semicircle limit that implies a bound of the form

$$\sup_{Imz \geq \eta} \left| \frac{1}{N} \sum_j \frac{1}{z - y_j} - \int \frac{\rho(r)\, dr}{z - r} \right| \leq \frac{1}{N^{\lambda/4}}$$

where η is of order $1/N^{1-\lambda}$ except on a set of probability exponentially small.

Previous result

(Erdös, Ramírez,Schlein, Yau) proved the semicircle law when the gaussian perturbation gets smaller with N. This was done for a perturbation of variance $N^{-3/4+\beta}$, for some $\beta > 0$.
The method used was based on entropy estimates from the OU process and orthogonal polynomials estimates inspired by ideas of Pastur and Scherbina.

Our result

(Erdös, Péché, Ramírez,Schlein, Yau) Let the distribution of the entries of H have a density $e^{-V(x)}$ with respect to gaussian measure. Assume that $V \in C^6$ and that

$$\sum_{j=1}^{6} |V^{(j)}(x)| \leq C(1+x^2)^k \quad \text{for some } k$$

and

$$\nu(x) = e^{-V(x)} e^{-x^2} \leq C e^{-\delta x^2}$$

Our result

Then

$$\int O(x_1, x_2)\frac{1}{N^2\rho^2}R_2^N(u+\frac{x_1}{\rho N}, u+\frac{x_2}{\rho N})dx_1\,dx_2 \to$$

$$\int O(x_1, x_2)\det\left(\frac{\sin\pi(ti-tj)}{\pi(ti-tj)}\right)_{i,j=1,2} dx_1\,dx_2$$

$(\rho = \rho(u) = \frac{2}{\pi}\sqrt{4-x^2})$

Other developments

At about the same time that this result was made public, Tao and Vu posted a similar result where the technical conditions were not needed but a matching of the third moments was necessary. Later, in a joint paper the methods were united to obtain an improved result (that still needed some decay of the distribution and averaging in the energy).

Other developments

Erdös, Yau, Schlein and Yin have continued working on the problem, eliminating most of the technical conditions and extending the result to generalized Wigner matrices. A deeper understanding of the phenomenon has also followed the new methods they have developed.

Proof: Stationary phase

First we analyze the asymptotics of a perturbation of a Wigner matrix by a GUE when the time of the OU is of order $t = 1/N^{1-\lambda}$.

Proof: Stationary phase

We use a formula for the kernel that is slightly different from Johansson's. For a fixed set of eigenvalues y of H one writes

$$R_2^N(x_1, x_2) = \int \det(K_N^S(x_1, x_2; y))_{i,j=1,2} \, dP^{(N)}(y)$$

Proof: Stationary phase

The kernel is given by

$$K_N^S(u,v;y) =$$

$$\frac{1}{(v-u)S(2\pi i)^2}\int_\gamma dz \int_\Gamma dw \left(e^{-(v-u)(w-r)/S}-1\right)\prod_j \frac{w-y_j}{z-y_j}$$

$$\frac{1}{w-r}\left(w-r+z-u-S\sum_j \frac{y_j-r}{(w-y_j)(z-y_j)}\right)$$

$$e^{(w^2-2uw-z^2+2uz)/2S}$$

where $r\in\mathbb{R}$, $\gamma=\{-s+i\omega : s\in\mathbb{R}\}\cup\{s-i\omega : s\in\mathbb{R}\}$ for any $\omega>0$ and $\Gamma=\{is : s\in\mathbb{R}\}$. Also $S=t/N$

Proof: Stationary phase

Let

$$f_N(z) = \frac{1}{2t}(z^2 - 2uz) + \frac{1}{N}\sum_j \log(z - y_j)$$

This funciton has two complex saddles that are conjugate to each other. They are to be denoted $q_N^{\pm}$. In fact, making use of the local semicircle law, one can see that

$$|q^{\pm} - q_N^{\pm}| \leq \frac{Ct}{N^{\lambda/4}}$$

where $q^{\pm}$ are the solutions to the corresponding limiting problem

$$q = u - 2t(q - \sqrt{q^2 - 1})$$

From here: $Imq \approx 2t\sqrt{1 - u^2} = t\rho/\pi$.

Proof: Stationary phase

The kernel can be written as

$$\frac{1}{N\rho} K_N(u, u + \frac{\tau}{N\rho}; y) =$$

$$N \int_\gamma \frac{dz}{2\pi i} \int_\Gamma \frac{dw}{2\pi i} h_N(w) g_N(z, w) e^{N(f_N(w) - f_N(z))}$$

where

$$h_N(w) = \frac{1}{\tau}\left(e^{-(w-r)/t\rho} - 1\right),$$

and g_N are well behaved functions, and

$$g_N(q_N^\pm, q_N^\pm) = f_N''(q_N^\pm)$$

Proof: Stationary phase

The paths can then be deformed so that they pass through the saddles and such that a Laplace asymptotic analysis is possible. This will give that the integral close to one saddle is approximately

$$\frac{-1}{2\pi f_N''(q_N^+)} g_N(q_N^+, q_N^+) h_N(q_N^+) e^{N(f_N(q_N^+) - f_N(q_N^+))}$$

$$= \frac{-1}{2\pi} h_N(q_N^+)$$

We take $r = Re\, q_N^+$.

Proof: Stationary phase

Given the different orientation of the integrals on the two saddles one has to substract to get the final answer:

$$\frac{1}{2\pi}\left[-h_N(q_N^+) + h_N(q_N^+)\right]$$

$$= \frac{1}{2\pi\tau}\left(-e^{-\tau Im(q_N^+)/t\rho} + e^{-\tau Im(q_N^-)/t\rho}\right) = \frac{\sin \pi\tau}{\pi\tau}$$

Proof: Reversed heat flow

To complete the proof, given a Wigner matrix H we need to find a matrix of the type $e^{tL}G$, where G is Wigner, such that H and $e^{tL}G$ are close when $t = 1/N^{1-\lambda}$. Note that we do not use $G = H$.

Proof: Reversed heat flow

Let F be the law of H. We require an ensemble of matrices G such that one can prove the universality result for $e^{Lt}G$ when $t \geq N^{-1+\lambda}$ and such that $e^{Lt}G$ and F are close.
Taking $G = e^{-Lt}F$ would solve the problem, if it was defined.
In its place we use

$$G_t = \left(1 - tL + \frac{1}{2}t^2L^2\right) F$$

Proof: Reversed heat flow

With that choice, and the regularity conditions on the measure, it is possible to prove that

$$\int \frac{|e^{tL}G_t - F|^2}{e^{tL}G_t} dGUE \leq CN^{-4+8\lambda}$$

This is enough to take the sine kernel result from G_t to F

The end

Thank you