Particle representations and limit theorems for stochastic partial differential equations

(or, if the only tool you have is a hammer. . .)

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A stochastic McKean-Vlasov equation

Kotelenez (1995); Kurtz and Xiong (1999)

Consider the stochastic partial differential equation

$$\begin{aligned}\langle V_\varepsilon(t),\varphi\rangle &= \langle V_\varepsilon(0),\varphi\rangle + \int_0^t \langle V_\varepsilon(s), L_\varepsilon\varphi(\cdot, V_\varepsilon(s))\rangle ds \\ &\quad + \int_{U\times[0,t]} \langle V_\varepsilon(s), \nabla\varphi(\cdot)^T \mathcal{J}_\varepsilon(\cdot, u, V_\varepsilon(s))\rangle W(du\times ds)\end{aligned}$$

where V_ε is measure-valued, $\langle \mu,\varphi\rangle = \int_{\mathbb{R}^d}\varphi(x)\mu(dx)$, W is space-time Gaussian white noise on $U\times[0,\infty)$ with variance measure $\nu(dy)\times ds$,

$$L_\varepsilon\varphi(x,\mu) = \frac{1}{2}\sum_{ij} D_{\varepsilon,ij}(x,\mu)\partial_i\partial_j\varphi(x) + F(x,\mu)\cdot\nabla\varphi(x)$$

$$D_\varepsilon(x,\mu) = \int_U \mathcal{J}_\varepsilon(x,u,\mu)\mathcal{J}_\varepsilon^T(x,u,\mu)\nu(du).$$

Note that $\langle V_\varepsilon(t),1\rangle \equiv \langle V_\varepsilon(0),1\rangle$; just assume $\langle V_\varepsilon,1\rangle\equiv 1$.

Exchangeability and de Finetti's theorem

$X_1, X_2, \ldots$ is *exchangeable* if

$$P\{X_1 \in \Gamma_1, \ldots, X_m \in \Gamma_m\} = P\{X_{s_1} \in \Gamma_1, \ldots, X_{s_m} \in \Gamma_m\}$$

$(s_1, \ldots, s_m)$ any permutation of $(1, \ldots, m)$.

Theorem 1 *(de Finetti) Let $X_1, X_2, \ldots$ be exchangeable. Then there exists a random probability measure Ξ such that for every bounded, measurable g,*

$$\lim_{n\to\infty} \frac{g(X_1) + \cdots + g(X_n)}{n} = \int g(x)\Xi(dx)$$

almost surely, and

$$E[\prod_{k=1}^{m} g_k(X_k)|\Xi] = \prod_{k=1}^{m} \int g_k d\Xi$$

Convergence of exchangeable systems

Kotelenez and Kurtz (2010)

Lemma 2 *For $n = 1, 2, \ldots$, let $\{\xi_1^n, \ldots, \xi_{N_n}^n\}$ be exchangeable (allowing $N_n = \infty$.) Let Ξ^n be the empirical measure (defined as a limit if $N_n = \infty$), $\Xi^n = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{\xi_i^n}$. Assume*

- $N_n \to \infty$
- *For each $m = 1, 2, \ldots$, $(\xi_1^n, \ldots, \xi_m^n) \Rightarrow (\xi_1, \ldots, \xi_m)$ in S^m.*

Then

$\{\xi_i\}$ is exchangeable and setting $\xi_i^n = s_0 \in S$ for $i > N_n$, $\{\Xi^n, \xi_1^n, \xi_2^n \ldots\} \Rightarrow \{\Xi, \xi_1, \xi_2, \ldots\}$ in $\mathcal{P}(S) \times S^\infty$, where Ξ is the deFinetti measure for $\{\xi_i\}$.

If for each m, $\{\xi_1^n, \ldots, \xi_m^n\} \to \{\xi_1, \ldots, \xi_m\}$ in probability in S^m, then $\Xi^n \to \Xi$ in probability in $\mathcal{P}(S)$.

Lemma 3 *Let* $X^n = (X_1^n, \ldots, X_{N_n}^n)$ *be exchangeable families of* $D_E[0,\infty)$*-valued random variables such that* $N_n \Rightarrow \infty$ *and* $X^n \Rightarrow X$ *in* $D_E[0,\infty)^\infty$*. Define*

$$\Xi_n = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{X_i^n} \in \mathcal{P}(D_E[0,\infty))$$

$$\Xi = \lim_{m\to\infty} \frac{1}{m}\sum_{i=}^{m} \delta_{X_i}$$

$$V_n(t) = \frac{1}{N_n}\sum_{i=1}^{N_n} \delta_{X_i^n(t)} \in \mathcal{P}(E)$$

$$V(t) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \delta_{X_i(t)}$$

Then

a) For $t_1, \ldots, t_l \notin \{t : E[\Xi\{x : x(t) \neq x(t-)\}] > 0\}$

$$(\Xi_n, V_n(t_1), \ldots, V_n(t_l)) \Rightarrow (\Xi, V(t_1), \ldots, V(t_l)).$$

b) If $X^n \Rightarrow X$ *in* $D_{E^\infty}[0,\infty)$*, then* $V_n \Rightarrow V$ *in* $D_{\mathcal{P}(E)}[0,\infty)$*. If* $X^n \to X$ *in probability in* $D_{E^\infty}[0,\infty)$*, then* $V_n \to V$ *in* $D_{\mathcal{P}(E)}[0,\infty)$ *in probability.*

Properties of cadlag processes

a) The set $D_\Xi = \{t : E[\Xi\{x : x(t) \neq x(t-)\}] > 0\}$ is at most countable.

b) If for $i \neq j$, with probability one, X_i and X_j have no simultaneous discontinuities, then $D_\Xi = \emptyset$ and convergence of X^n to X in $D_E[0,\infty)^\infty$ implies convergence in $D_{E^\infty}[0,\infty)$.

From particle approximation to particle representation

Let $X_\varepsilon^N = \{X_{\varepsilon,i}^N\}$ satisfy

$$X_{\varepsilon,i}^N(t) = X_{\varepsilon,i}^N(0) + \int_{U\times[0,t]} \mathcal{J}_\varepsilon(X_{\varepsilon,i}^N(s), u, V_\varepsilon^N(s))\rangle W(du \times ds) + \int_0^t F(X_{\varepsilon,i}^N(s), V_\varepsilon^N(s))ds$$

where $V_\varepsilon^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{X_{\varepsilon,i}^N(t)}$ and $\{X_{\varepsilon,i}^N(0), 1 \le i \le N)\}$ is exchangeable.

Assume

$$(x,\mu) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to F(x,\mu) \in \mathbb{R}^d$$

and

$$(x,\mu) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathcal{J}_\varepsilon(x,\cdot,\mu) \in L^2(\nu)$$

are bounded and continuous. Then there exists an exchangeable (weak) solution. (Construct an Euler approximation and pass to the limit.)

Convergence to infinite system

Letting $N \to \infty$ and applying Lemma 3, there exists an exchangeable solution to the infinite system

$$X_{\varepsilon,i}(t) = X_{\varepsilon,i}(0) + \int_{U\times[0,t]} \mathcal{J}_\varepsilon(X_{\varepsilon,i}(s), u, V_\varepsilon(s))\rangle W(du \times ds) + \int_0^t F(X_{\varepsilon,i}(s), V_\varepsilon(s))ds$$

where $V_\varepsilon(t)$ is the de Finetti measure of $\{X_{\varepsilon,i}(t)\}$.

Uniqueness for infinite system

Kurtz and Protter (1996); Kurtz and Xiong (1999)

Theorem 4 *Let*

$$\rho(\mu_1, \mu_2) = \sup_{\{f:|f(x)-f(y)|\leq|x-y|\}} |\int_{\mathbb{R}^d} f d\mu_1 - \int_{\mathbb{R}^d} f d\mu_2|$$

and assume

$$|F(x_1, \mu_1) - F(x_2, \mu_2)| + \|\mathcal{J}_\varepsilon(x_1, \cdot\mu_1) - \mathcal{J}_\varepsilon(x_2, \cdot, \mu_2)\|_{L^2(\nu)} \leq C(|x_1 - x_2| + \rho(\mu_1, \mu_2)).$$

Then the solution of the infinite system is unique.

The corresponding SPDE

$$\begin{aligned}\varphi(X_{\varepsilon,i}(t)) &= \varphi(X_{\varepsilon,i}(0)) + \int_0^t L_\varepsilon\varphi(X_{\varepsilon,i}(s), V_\varepsilon(s))ds \\ &\quad + \int_{U\times[0,t]} \nabla\varphi(X_{\varepsilon,i}(s))^T \mathcal{J}_\varepsilon(X_{\varepsilon,i}(s), u, V_\varepsilon(s)) W(du \times ds)\end{aligned}$$

By the exchangeablity, averaging over i gives

$$\begin{aligned}\langle V_\varepsilon(t), \varphi\rangle &= \langle V_\varepsilon(0), \varphi\rangle + \int_0^t \langle V_\varepsilon(s), L_\varepsilon\varphi(\cdot, V_\varepsilon(s))\rangle ds \\ &\quad + \int_{U\times[0,t]} \langle V_\varepsilon(s), \nabla\varphi(\cdot)^T \mathcal{J}_\varepsilon(\cdot, u, V_\varepsilon(s))\rangle W(du \times ds)\end{aligned}$$

Uniqueness via Markov mapping

Define $\gamma : (\mathbb{R}^d)^\infty \to \mathcal{P}(\mathbb{R}^d)$ by

$$\gamma(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}$$

if the limit exists in $\mathcal{P}(\mathbb{R}^d)$ and $\gamma(x) = \mu_0$ otherwise.

Then $V_\varepsilon(t) = \gamma(X_\varepsilon(t))$ and a Markov mapping theorem implies that every solution of the SPDE can be obtained in this way.

Vanishing spatial noise correlations

Let $d \geq 2$, $U = \mathbb{R}^d$, ν be Lebesgue measure, and

$$\mathcal{J}_\varepsilon(x, u, \mu) = \varepsilon^{-d/2}\mathcal{J}(x, \varepsilon^{-1}(x-u), \mu),$$

so that the stochastic intergral becomes

$$\begin{aligned}
&\int_{\mathbb{R}^d\times[0,t]} \mathcal{J}_\varepsilon(X_{\varepsilon,i}(s), u, V_\varepsilon(s))\rangle W(du \times ds) \\
&\quad = \int_{\mathbb{R}^d\times[0,t]} \varepsilon^{-d/2}\mathcal{J}(X_{\varepsilon,i}(s), \varepsilon^{-1}(X_{\varepsilon,i}(s) - u), V_\varepsilon(s))\rangle W(du \times ds) \\
&\quad = \int_{\mathbb{R}^d\times[0,t]} \mathcal{J}(X_{\varepsilon,i}(s), z, V_\varepsilon(s))\rangle W_i^\varepsilon(dz \times ds),
\end{aligned}$$

where for each i, W_i^ε is a Gaussian white noise defined by

$$\int_{\mathbb{R}^d\times[0,\infty)} \varphi(z,s)W_i^\varepsilon(dz\times ds) = \int_{\mathbb{R}^d\times[0,t]} \varepsilon^{-d/2}\varphi(\varepsilon^{-1}(X_{\varepsilon,i}(s)-u), s)\rangle W(du\times ds)$$

(NOTE: The W_i^ε are not independent but are exchangeable.)

Convergence

If $\int_{\mathbb{R}^d} |\mathcal{J}(x,z,\mu)|^2 dz < \infty$ and $x_1 \neq x_2$, then

$$\begin{aligned}\int_{\mathbb{R}^d} & \varepsilon^{-d/2}\mathcal{J}(x_1,\varepsilon^{-1}(x_1-u),\mu)\varepsilon^{-d/2}\mathcal{J}(x_2,\varepsilon^{-1}(x_2-u),\mu)^T du \\ &= \int_{\mathbb{R}^d} \mathcal{J}(x_1,\varepsilon^{-1}x_1-u),\mu)\mathcal{J}(x_2,\varepsilon^{-1}x_2-u),\mu)^T du \\ &\to 0\end{aligned}$$

Assume that the convergence is uniform on $|x_1 - x_2| \geq \delta > 0$, for each $\delta > 0$, and on compact subsets of $\mathcal{P}(\mathbb{R}^d)$.

Assume the nondegeneracy condition

$$\inf_{x,\mu}\inf_{z} \frac{\int_{\mathbb{R}^d}(z\cdot\mathcal{J}(x,z,\mu))^2 du}{|z|^2} > 0.$$

The zero correlation limit

Theorem 5 *Assume $X_\varepsilon(0) = X(0)$, and an additional regularity condition for $d = 2$. As $\varepsilon \to 0$, X_ε converges in distribution to the solution of*

$$X_i(t) = X_i(0) + \int_{\mathbb{R}^d \times [0,t]} \mathcal{J}(X_i(s), u, V(s))\rangle W_i(du \times ds) + \int_0^t F(X_i(s), V(s))ds$$

where the W_i are independent and $V(t)$ is the de Finetti measure for $\{X_i\}$.

V is the unique solution of

$$\langle V(t), \varphi \rangle = \langle V(0), \varphi \rangle + \int_0^t \langle V(s), L\varphi(\cdot, V(s))\rangle ds$$

where $L\varphi(x, \mu) = \frac{1}{2} \sum_{ij} D_{ij}(x, \mu)\partial_i \partial_j \varphi(x) + F(x, \mu) \cdot \nabla \varphi(x)$

$$D(x, \mu) = \int_U \mathcal{J}(x, u, \mu)\mathcal{J}(x, u, \mu)^T du.$$

Sketch of proof

$$\begin{aligned} X_{\varepsilon,i}(t) &= X_i(0) + \int_{U\times[0,t]} \varepsilon^{-d/2}\mathcal{J}(X_{\varepsilon,i}(s), \frac{X_{\varepsilon,i}-u}{\varepsilon}, V_\varepsilon(s))\rangle W(du\times ds) \\ &\quad + \int_0^t F(X_{\varepsilon,i}(s), V_\varepsilon(s))ds \\ &= X_i(0) + \int_{\mathbb{R}^d\times[0,t]} \mathcal{J}(X_{\varepsilon,i}(s), u, V_\varepsilon(s))\rangle W_i^\varepsilon(du\times ds) + \int_0^t F(X_{\varepsilon,i}(s), V_\varepsilon(s))ds \end{aligned}$$

where

$$M_i^{\varphi,\varepsilon}(t) = \int_{\mathbb{R}^d\times[0,t]} \varphi(u) W_i^\varepsilon(du\times ds) = \int_{\mathbb{R}^d\times[0,t]} \varepsilon^{-d/2}\varphi(\frac{X_{\varepsilon,i}(s)-u}{\varepsilon})W(du\times ds)$$

Relative compactness follows from boundedness of $\mathcal{J}$ and F and

$$[M_i^{\varphi,\varepsilon}, M_j^{\psi,\varepsilon}]_t = \int_{\mathbb{R}^d\times[0,t]} \varepsilon^{-d/2}\varphi(\frac{X_{\varepsilon,i}(s)-u}{\varepsilon})\varepsilon^{-d/2}\psi(\frac{X_{\varepsilon,j}(s)-u}{\varepsilon})du \to 0$$

for $t < \tau_{ij} = \inf\{t : X_i(t) = X_j(t)\}$. If $\tau_{ij} = \infty$ a.s., then the limits W_i and W_j are independent.

A stochastic Allen-Cahn equation

Bertini, Brassesco, and Buttà (2009)

$$dm = (\frac{1}{2}m_{xx} - (m^2-1)m)dt + c\dot{W}, \quad m(t,b) = m_+, m(t,a) = m_-$$

Find a signed measure-valued process M which will have a density with respect to Lebesgue measure so we can write $M(t,dx) = \overline{M}(t,x)dx$.

Let $\{X_i\}$ be independent reflecting Browian motions on the interval $[a,b]$ with uniform initial distribution, and let A_i satisfy

$$\begin{aligned} A_i(t) = A_i(0) + \int_0^t G(\overline{M}(s,X_i(s))A_i(s)ds + \int_{\mathbb{R}\times[0,t]} \rho_\epsilon(X_i(s)-u)W(du\times ds) \\ + \int_0^t \frac{m_+ - A_i(s-)}{|m_+ - A_i(s-)|}d\Lambda_i^+(s) + \int_0^t \frac{m_- - A_i(s-)}{|m_- - A_i(s-)|}d\Lambda_i^-(s), \end{aligned}$$

where

$$\langle M(t), \varphi\rangle = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \varphi(X_i(t))A_i(t).$$

Assumptions

- $\{A_i(0)\}$ is an exchangeable sequence.
- W is space-time Gaussian white noise governed by Lebesgue measure so $\int_{\mathbb{R}\times[0,t]} \rho_\epsilon(X_i(s)-u)W(du\times ds)$ is a Brownian motion with variance $t\int_{\mathbb{R}} \rho_\epsilon(u)^2 du$.
- Λ_i^+ and Λ_i^- are right continuous, nondecreasing processes, Λ_i^+ increases only when $X_i = b$ and Λ_i^- increases only when $X_i = a$, and if $X_i(s) = b$, $\Lambda_i^+(s) - \Lambda_i^+(s-) = |m_+ - A_i(s-)|$ and if $X_i(s) = b$, $\Lambda_i^-(s) - \Lambda_i^-(s-) = |m_- - A_i(s-)|$.
- Note that if $X_i(s) = b$, then $A_i(s) = m_+$ and if $X_i(s) = a$, $A_i(s) = m_-$.
- We require the solution $\{(X_i, A_i)\}$ to be exchangeable.

Derivation of SPDE

M is the signed measure given by

$$\langle \varphi, M(t) \rangle = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(X_i(t)) A_i(t)$$

and $\overline{M}$ is the density of M.

Note that the limit will exist by the exchangeability requirement and M is absolutely continuous with respect to Lebesgue measure by the uniformity of the $\{X_i(t)\}$.

Let X_i be given by the Skorohod equation

$$X_i(t) = X_i(0) + \sigma B_i(t) + \lambda_i^a(t) - \lambda_i^b(t),$$

where the B_i are independent standard Browian motions independent of W and $\{X_i(0)\}$, the $X_i(0)$ are independent and uniformly distributed over $[a, b]$, and λ_i^a and λ_i^b are the local times at a and b.

For $\varphi \in C_c^2(a,b)$,

$$\begin{aligned}\varphi(X_i(t))A_i(t) &= \varphi(X_i(0))A_i(0) + \int_0^t \varphi(X_i(s))dA_i(s) \\ &\quad + \int_0^t \varphi'(X_i(s))A_i(s)dB_i(s) + \int_0^t \frac{1}{2}\varphi''(X_i(s))A_i(s)ds \\ &= \varphi(X_i(0))A_i(0) + \int_0^t \varphi(X_i(s))G(\overline{M}(s,X_i(s)))A_i(s)ds \\ &\quad + \int_{\mathbb{R}\times[0,t]} \varphi(X_i(s))\rho_\epsilon(X_i(s)-u)W(du\times ds) \\ &\quad + \int_0^t \varphi'(X_i(s))A_i(s)dB_i(s) + \int_0^t \frac{1}{2}\varphi''(X_i(s))A_i(s)ds\end{aligned}$$

Averaging both sides of this identity,

$$\begin{aligned}\langle \varphi, M(t)\rangle &= \langle \varphi, M(0)\rangle + \int_0^t \langle \varphi G(\overline{M}(s,\cdot)), M(s)\rangle ds \\ &\quad + \int_{\mathbb{R}\times[0,t]}\int_{\mathbb{R}} \varphi(x)\rho_\epsilon(x-u)dxW(du\times ds) + \int_0^t \langle \frac{1}{2}\varphi'', M(s)\rangle ds\end{aligned}$$

subject to the boundary conditions $\overline{M}(t,b) = m_+$ and $\overline{M}(t,a) = m_-$.

What happens when ϵ goes to zero?

Let ϵ be the diameter of the support of ρ_ϵ.

$$\int_{\mathbb{R}\times[0,t]}\int_{\mathbb{R}}\varphi(x)\rho_\epsilon(x-u)dxW(du\times ds)$$

converges to

$$c\int_{\mathbb{R}\times[0,t]}\int_{\mathbb{R}}\varphi(u)W(du\times ds)$$

if $\int_{\mathbb{R}}\rho_\epsilon(x)dx \to c$.

$$W_i^\epsilon(t)=\int_{\mathbb{R}\times[0,t]}\rho_\epsilon(X_i(s)-u)W(du\times ds)$$

converge to independent Brownian motions with parameter σ^2 if

$$\int_{\mathbb{R}}\rho_\epsilon^2(x)dx \to \sigma^2.$$

Conditionally Poisson representations

Kurtz and Rodrigues (2010)

Consider

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s))dB_i(s) + \int_0^t c(X_i(s))ds + \int_{\Gamma\times[0,t]} \widehat{\sigma}(X_i(s),u)W(du\times ds)$$

and

$$U_i(t) = U_i(0) + \int_0^t U_i(s)\gamma_0(X_i(s))dB_i(s) - \int_0^t U_i(s)d(X_i(s))ds + \int_{\Gamma\times[0,t]} U_i(s)\gamma_1(X_i(s),u)W(du\times ds)$$

B_i independent Brownian motions,

$\{(X_i(0),U_i(0))\}$ a conditionally Poisson point process with mean measure $V(0,dx)\times du$.

Measure-valued process

$$
\begin{aligned}
V(t) &= \lim_{r\to\infty}\frac{1}{r}\sum_{U_i(t)\le r}\delta_{X_i(t)} = \lim_{\epsilon\to 0}\epsilon\sum_i e^{-\epsilon U_i(t)}\delta_{X_i(t)} \\
&= \lim_{\epsilon\to\infty}\epsilon^2\sum_i U_i(t)e^{-\epsilon U_i(t)}\delta_{X_i(t)} = \frac{1}{2}\epsilon^3\sum_i e^{-\epsilon U_i(t)}U_i^2(t)\delta_{X_i(t)}
\end{aligned}
$$

Define

$$
L_1\varphi = \frac{1}{2}a\varphi'' + c\varphi', \quad L_2\varphi = d\varphi - (\gamma_0\sigma + \int_\Gamma \gamma_1\widehat{\sigma}d\mu)\varphi'
$$

$$
\beta(x) = \gamma_0(x)^2 + \int_\Gamma \gamma_1^2(x,u)\mu(du)
$$

Applying Itô's formula

$$
\begin{aligned}
&e^{-\epsilon U_i(t)}\varphi(X_i(t)) \\
&\quad = e^{-\epsilon U_i(0)}\varphi(X_i(0)) - \epsilon \int_0^t U_i(s) e^{-\epsilon U_i(s)} \varphi(X_i(s))\gamma_0(X_i(s))dB_i(s) \\
&\qquad + \int_0^t e^{-\epsilon U_i(s)}\sigma(X_i(s))\varphi'(X_i(s))dB_i(s) \\
&\qquad - \epsilon \int_{\Gamma\times[0,t]} U_i(s) e^{-\epsilon U_i(s)}\varphi(X_i(s))\gamma_1(X_i(s),u)W(du\times ds) \\
&\qquad + \int_{\Gamma\times[0,t]} e^{-\epsilon U_i(s)}\widehat{\sigma}(X_i(s),u)\varphi'(X_i(s))W(du\times ds) \\
&\qquad + \frac{1}{2}\epsilon^2 \int_0^t e^{-\epsilon U_i(s)} U_i^2(s)\varphi(X_i(s))\beta(X_i(s))ds \\
&\qquad + \int_0^t e^{-\epsilon U_i(s)} L_1\varphi(X_i(s))ds + \epsilon \int_0^t U_i(s) e^{-\epsilon U_i(s)} L_2\varphi(X_i(s))ds
\end{aligned}
$$

Corresponding SPDE

$$\begin{aligned}\langle V(t),\varphi\rangle &= \langle V(0),\varphi\rangle - \int_{\Gamma\times[0,t]}\langle V(t),(\gamma_1(\cdot,u)\varphi+\widehat{\sigma}(\cdot,u)\varphi')\rangle W(du\times ds)\\ &\quad + \int_0^t \langle V(t),L\varphi\rangle ds\end{aligned}$$

$$L\varphi = \frac{1}{2}a\varphi'' + (c-\gamma_0\sigma-\int_\Gamma \gamma_1\widehat{\sigma}d\mu)\varphi' + (d+\beta)\varphi$$

For the Zakai equation, take

$$\widehat{\sigma}=\gamma_0=0,\quad d=-\beta$$

Markov mapping theorem

Theorem 6 *$A \subset \overline{C}(E) \times \overline{C}(E)$ a pre-generator with bp-separable graph.*

$\mathcal{D}(A)$ closed under multiplication and separating.

$\gamma : E \to E_0$, Borel measurable.

α a transition function from E_0 into E satisfying

$$\alpha(y, \gamma^{-1}(y)) = 1$$

Let $\mu_0 \in \mathcal{P}(E_0)$, $\nu_0 = \int \alpha(y, \cdot)\mu_0(dy)$, and define

$$C = \{(\int_E f(z)\alpha(\cdot, dz), \int_E Af(z)\alpha(\cdot, dz)) : f \in \mathcal{D}(A)\} .$$

If $\widetilde{Y}$ is a solution of the MGP for (C, μ_0), then there exists a solution Z of the MGP for (A, ν_0) such that $Y = \gamma \circ Z$ and $\widetilde{Y}$ have the same distribution on $M_{E_0}[0, \infty)$.

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Abstract

Particle representations and limit theorems for stochastic partial differential equations

Solutions of the a large class of stochastic partial differential equations can be represented in terms of the de Finetti measure of an infinite exchangeable system of stochastic ordinary differential equations. These representations provide a tool for proving uniqueness, obtaining convergence results, and describing properties of solutions of the SPDEs. The basic tools for working with the representations will be described. Applications include the convergence of an SPDE as the spatial correlation length of the noise vanishes, uniqueness for a class of SPDEs, and consistency of approximation methods for the classical filtering equations.