Small time asymptotics for fast mean-reverting stochastic volatility models

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Background

Let S_t denote stock price at time t.

A *European call option* on this stock is a contract that gives its holder the right, but not the obligation to buy a unit of this stock at a certain price, called *strike price*, and at a given time called *maturity time*.

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A *European call option* on this stock is a contract that gives its holder the right, but not the obligation to buy a unit of this stock at a certain price, called *strike price*, and at a given time called *maturity time*.

Let $K = strike\ price$, $T = maturity\ time$ and suppose $S_0 < K$ (out-of-the- money).

Question

What is the behavior of

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To estimate this quantity, we study

$$P(S_T > K)$$

as $T \to 0$.
This probability decays exponentially fast to 0. We get a large deviation estimate of the form

$$\lim_{T \to 0} T \log P(S_T > K) = -I(K)$$

Black-Scholes model

A simple model for this stock price is the B-S model

$$dS_t = rS_t dt + \sigma S_t dW_t;$$

$\sigma > 0$ is called *volatility*.
Under the B-S model, the price of a call option with strike price K and maturity time T is:

$$option\ price = E[e^{-rT}(S_T - K)^+]$$

is easy to calculate.

However the assumption of constant volatility is unrealistic and we instead work with a more sophisticated model.

Stochastic volatility model for stock price

Let S_t denote stock price.

$$dS_t = rS_t dt + S_t \sigma(Y_t) dW_t^{(1)}, \tag{1a}$$

$$dY_t = \frac{1}{\delta}(m - Y_t)dt + \frac{\nu}{\sqrt{\delta}} Y_t^{\beta} dW_t^{(2)}. \tag{1b}$$

where $m \in \mathbb{R}, r, \nu > 0$, $W^{(1)}$ and $W^{(2)}$ are standard Brownian motions with $\langle W^{(1)}, W^{(2)} \rangle_t = \rho t$, with $|\rho| < 1$ constant.

-The process (Y_t) is a fast mean-reverting process with rate of mean reversion $1/\delta$ $(\delta > 0)$.

Assumptions

We assume that

Assumption

1. $\beta \in \{0\} \cup [\frac{1}{2}, 1)$;
2. *in the case of* $\beta = 1/2$, *we require* $m > \nu^2/2$ *and* $Y_0 > 0$ *a.s., in the case of* $1/2 < \beta < 1$, *we require* $m > 0$ *and* $Y_0 > 0$ *a.s.;*
3. $\sigma(y) \in C(\mathbb{R}; \mathbb{R}_+)$ *satisfies*

$$\sigma(y) \leq C(1 + |y|^{\sigma}),$$

for some constants $C > 0$ *and* σ *with* $0 \leq \sigma < 1 - \beta$.

Examples of Y process

- **Ornstein-Uhlenbeck process** (take $\beta = 0$)

$$dY_t = \frac{1}{\delta}(m - Y_t)dt + \frac{\nu}{\sqrt{\delta}} dW_t^{(2)}.$$

- **CIR process** (take $\beta = 1/2$)

$$dY_t = \frac{1}{\delta}(m - Y_t)dt + \frac{\nu}{\sqrt{\delta}} \sqrt{Y_t} dW_t^{(2)}.$$

Rescaling time

Let $X_t = \log S_t$. Rescale time $t \mapsto \epsilon t$.

$$dX_{\epsilon,t} = \epsilon \left(r - \frac{1}{2}\sigma^2(Y_{\epsilon,t}) \right) dt + \sqrt{\epsilon}\,\sigma(Y_{\epsilon,t}) dW_t^{(1)} \tag{2a}$$

$$dY_{\epsilon,t} = \frac{\epsilon}{\delta}(m - Y_{\epsilon,t})dt + \nu\sqrt{\frac{\epsilon}{\delta}}\, Y_{\epsilon,t}^{\beta} dW_t^{(2)}. \tag{2b}$$

Our mean-reversion time δ is ϵ-dependent.

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- $\delta = \epsilon^4$ (ultra-fast regime)
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- $\delta = \epsilon^2$ (fast regime)

As $\epsilon \rightarrow 0$, we look at small time asymptotics of X process but large time asymptotics of the Y process. Y is mean-reverting and ergodic and approaches its invariant distribution in large time. The effect of Y gets averaged!

Large Deviation Principle (LDP)

Let $X_{\epsilon,0} = x$. We prove the following large deviation estimates of the probabilities of $\{X_{\epsilon,t} > x'\}$ when $x' > x$.

THEOREM

$$\lim_{\epsilon \longrightarrow 0} \epsilon \log P(X_{\epsilon,t} > x') = -I(x'; x, t)$$

with rate functions $I(x'; x, t)$ as follows.

Rate function ($\delta = \epsilon^4$ case)

When $\delta = \epsilon^4$,

$$I(x'; x, t) = \frac{|x - x'|^2}{2\bar{\sigma}^2 t},$$

where $\bar{\sigma}^2$ is the average of the volatility function $\sigma(y)$ with respect to the invariant distribution of Y.

Rate function ($\delta = \epsilon^2$ case)

When $\delta = \epsilon^2$,

$$I(x'; x, t) = t \sup_{p \in \mathbb{R}} \left\{ p\left(\frac{x - x'}{t}\right) - \bar{H}_0(p) \right\}$$

where

$$\bar{H}_0(p) = \lim_{T \to +\infty} T^{-1} \log E[e^{\frac{1}{2}p^2 \int_0^T \sigma^2(Y_s^p) ds} | Y_0^p = y].$$

Y^p is the process with the perturbed Y process with generator B^p

$$B^p g(y) = Bg(y) + \rho\sigma\nu y^\beta p \partial_y g(y), \quad (3)$$

where B is the generator of the Y process.

Problem

Two aspects to this problem:

It is a Large Deviation problem coupled with a homogenization problem.

Key Steps in Proof

- Prove convergence of the following functionals

$$u_\epsilon^h(t,x,y) := \epsilon \log E[e^{\epsilon^{-1}h(X_{\epsilon,t})}|X_{\epsilon,0} = x, Y_{\epsilon,0} = y], \quad h \in C_b(\mathbb{R})$$

to $u_0^h(t,x)$.

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  to $u_0^h(t,x)$.
- Prove exponential tightness of $\{X_{\epsilon,t}\}_{\epsilon>0}$, i.e. for any $\alpha > 0$ there exists a compact set $K_\alpha \subset \mathbb{R}$ such that

$$\lim_{\epsilon \to 0} \epsilon \log P(X_{\epsilon,t} \notin K_\alpha) \leq -\alpha.$$

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Then, by Bryc's inverse Varadhan lemma, $\{X_{\epsilon,t}\}_{\epsilon>0}$ satisfies a LDP with

$$I(x';x,t) := \sup_{h \in C_b(\mathbb{R})} \{h(x') - u_0^h(t,x)\}.$$

Convergence of u_ϵ

Fix $h \in C_b(\mathbb{R})$. How do we prove

$$u_\epsilon(t,x,y) := \epsilon \log E[e^{\epsilon^{-1} h(X_{\epsilon,t})} | X_{\epsilon,0} = x, Y_{\epsilon,0} = y]$$

converges?

Convergence of u_ϵ

Possible ways:

- Compute u_ϵ and take limit.
- Use PDE (viscosity solution) approach.
- Operator-theoretic approach (See Feng and Kurtz [2]):

$$u_\epsilon^h = S_\epsilon(t)h$$

  is a nonlinear contraction semigroup.
  Method: Let H_ϵ denote nonlinear generator of $S_\epsilon(t)$. Prove $H_\epsilon \to H$. Invoke Crandall-Liggett generation theorem to get the limit semigroup $S(t)$ corresponding to H.
- A variational representation for $S_\epsilon(t)h$ can be obtained. $S_\epsilon(t)$ can be interpreted as a Nisio semigroup.

PDE

u_ϵ satisfies the following nonlinear pde:

$$\partial_t u = H_\epsilon u, \quad \text{in } (0, T] \times \mathbb{R} \times E_0; \tag{4a}$$

$$u(0, x, y) = h(x), \quad (x, y) \in \mathbb{R} \times E_0. \tag{4b}$$

E_0 is the state space of Y.

H_ϵ

Let A_ϵ denote the generator of the markov process $(X_{\epsilon,\cdot}, Y_{\epsilon,\cdot})$. Then

$$\begin{aligned} H_\epsilon u(t,x,y) &= \epsilon e^{-\epsilon^{-1}u} A_\epsilon e^{\epsilon^{-1}u}(t,x,y) \\ &= \epsilon\Big((r - \frac{1}{2}\sigma^2)\partial_x u + \frac{1}{2}\sigma^2\partial^2_{xx} u\Big) + \frac{1}{2}|\sigma\partial_x u|^2 \\ &\quad + \frac{\epsilon^2}{\delta} e^{-\epsilon^{-1}u} B e^{\epsilon^{-1}u} + \rho\sigma(y)\nu y^\beta\big(\frac{\epsilon}{\sqrt{\delta}}\partial^2_{xy} u + \frac{1}{\sqrt{\delta}}\partial_x u \partial_y u\big) \end{aligned} \tag{5}$$

where,

$$\frac{\epsilon^2}{\delta} e^{-\epsilon^{-1}u} B e^{\epsilon^{-1}u} = \frac{\epsilon}{\delta} Bu + \delta^{-1}\frac{1}{2}|\nu y^\beta \partial_y u|^2.$$

u_0 for $\delta = \epsilon^4$ case

We prove that as $\epsilon \to 0$, $u_\epsilon \to u_0$ where u_0 satisfies the HJB equation

$$\partial_t u_0 = \frac{1}{2}|\bar{\sigma}\partial_x u_0(x)|^2;$$
$$u_0(0, x) = h(x),$$

where $\bar{\sigma}^2$ is the average of $\sigma^2(\cdot)$ with respect to the invariant distribution of Y.

u_0 for $\delta = \epsilon^2$ case

We prove that as $\epsilon \longrightarrow 0$, $u_\epsilon \longrightarrow u_0$ where u_0 satisfies the HJB equation

$$\partial_t u_0 = \bar{H}_0(\partial_x u_0)$$
$$u_0(0,x) = h(x),$$

where

$$\bar{H}_0(p) = \lim_{T\longrightarrow+\infty} T^{-1} \log E[e^{\frac{1}{2}p^2 \int_0^T \sigma^2(Y_s^p)ds} | Y_0^p = y].$$

Rigorous proof of convergence of u_ϵ

We use viscosity solution techniques adapted from Feng and Kurtz [2].
Difficulties:

- In what sense do the operators H_ϵ converge? How do we identify the limit operator?
- We are averaging over a non-compact space!

-We need to carefully choose suitable perturbed test functions in our proof.
-The perturbed test functions $f_\epsilon(t,x,y)$ and $H_\epsilon f_\epsilon(t,x,y)$ should have compact level sets.

Rigorous proof of convergence of u_ϵ

We prove conditions for convergence of u_ϵ, solutions (in the viscosity solution sense) of

$$\partial_t u = H_\epsilon u,$$

to a sub-solution of

$$\partial_t u(t,x) \le \inf_{\alpha\in\Lambda} H_0(x, \nabla u(t,x), D^2 u(t,x); \alpha), \tag{8}$$

and a super-solution of

$$\partial_t u(t,x) \ge \sup_{\alpha\in\Lambda} H_1(x, \nabla u(t,x), D^2 u(t,x); \alpha). \tag{9}$$

The method used is a generalization of Barles-Perthame's half-relaxed limit arguments first introduced in single scale, compact space setting (see Fleming and Soner [3]).

Asymptotics of option price

Consider an out-of-the-money European call option i.e.

$$S_0 < K$$

or

$$x = \log S_0 < \log K.$$

Lemma

$$\lim_{\epsilon \longrightarrow 0^+} \epsilon \log E[e^{-r\epsilon t}(S_{\epsilon,t} - K)^+] = -I(\log K; x, t),$$

where I is the rate function for LDP of $\{X_{\epsilon,t}\}_{\epsilon>0}$.

Asymptotics of option price

Proof.

$$\begin{aligned}\lim_{\epsilon\to0^+}\epsilon\log E[e^{-r\epsilon t}(S_{\epsilon,t}-K)^+] &= \lim_{\epsilon\to0^+}\epsilon\log\int_K^\infty P(S_{\epsilon,t}>z)dz\\ &= \lim_{\epsilon\to0^+}\epsilon\log\int_K^\infty P(X_{\epsilon,t}>\log z)dz\\ &\approx \lim_{\epsilon\to0^+}\epsilon\log\int_K^\infty \exp\left\{-\frac{I(\log z;x,t)}{\epsilon}\right\}dz\\ &= -\inf_{z\in(K,\infty)} I(\log z;x,t)\end{aligned}$$

(by Laplace principle)

$$= -I(\log K;x,t)$$

as I is a continuous, increasing function in the interval (x,∞). □

Details of this work can be found on http://arxiv.org/abs/1009.2782.

Our paper titled "*Small time asymptotics for fast mean-reverting stochastic volatility models*" has been accepted (pending minor revisions) in Annals of Applied Probability.

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Thank You!

Black-Scholes model

A simple model for this stock price is the B-S model

$$dS_t = rS_t dt + \sigma S_t dW_t;$$

$\sigma > 0$ is called *volatility*.
Price of a call option with strike price K and maturity time T under this model can be easily calculated:

$$E^{BS}\left[e^{-rT}(S_T - K)^+\right]$$
$$= S_0 \Phi\left(\frac{\log(S_0/K) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log(S_0/K) + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$

(Black-Scholes Formula)

Implied Volatility

Implied Volatility, $\Sigma(T, K)$, is the volatility parameter value to be inputed in the Black-Scholes model to match a call option price.

Implied volatility for Black-Scholes model is a constant for all T and K.

However, implied volatilities of market prices are *not constant* and vary with T and K. Keeping T fixed, the graph of implied volatilities of market prices as a function of K is approximately U-shaped.

Implied volatility

Let σ_ϵ denote the implied volatility corresponding to strike price K of option price given by our stochastic volatility model. Then σ_ϵ is obtained by solving:

$$e^{r\epsilon t} S_0 \Phi\left(\frac{x - \log K + r\epsilon t + \frac{1}{2}\sigma_\epsilon^2 \epsilon t}{\sigma_\epsilon \sqrt{\epsilon t}}\right) - K\Phi\left(\frac{x - \log K + r\epsilon t - \frac{1}{2}\sigma_\epsilon^2 \epsilon t}{\sigma_\epsilon \sqrt{\epsilon t}}\right)$$

$$= E\left[e^{-r\epsilon t}(S_{\epsilon,t} - K)^+\right] \approx e^{-\frac{I(\log K; x, t)}{\epsilon}}$$

Taking $\lim_{\epsilon \to 0} \epsilon \log$ on both sides, we get

$$\lim_{\epsilon \to 0^+} \sigma_\epsilon^2 = \frac{(\log K - x)^2}{2I(\log K; x, t)t}.$$

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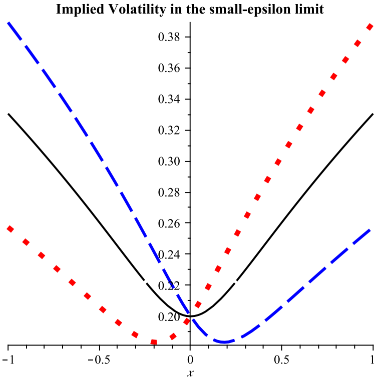
In the regime $\delta = \epsilon^4$, we get $\lim_{\epsilon \to 0^+} \sigma_\epsilon^2 = \bar{\sigma}^2$.

In the regime $\delta = \epsilon^2$, we need to first compute the quantity $\bar{H}_0$ defined as the limit of a log moment. This can be computed for the Heston model i.e. when $\sigma(y) = \sqrt{y}$ and $\beta = 1/2$.

A LDP and applications to option pricing and implied volatility for the Heston Model are in Feng, Forde and Fouque [1].



Here $x = \log(K/S_0)$. Take $\sigma(y) = \sqrt{y}$ and the parameters are $t = 1, \beta = 1/2, m = .04, \nu = 1.74$ and $\rho = -.4$ (dashed blue), $\rho = 0$ (solid black), $\rho = +.4$ (dotted red).

Rate functions ($\delta = \epsilon^4$)

$$I(x'; x, t) = \frac{|x' - x|^2}{2\bar{\sigma}^2 t}$$

is the rate function for $\{Z_{\epsilon,t}\}_{\epsilon>0}$ where $Z_{\epsilon,\cdot}$ satisfies $Z_{\epsilon,0} = x$ and

$$Z_{\epsilon,t} = x + \epsilon \int_0^t \left(r - \frac{1}{2}\bar{\sigma}^2 \right) ds + \sqrt{\epsilon} \int_0^t \bar{\sigma} dW_s$$

$(Z_{\epsilon,t} = \log S^{BS}_{\epsilon,t})$

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($Z_{\epsilon,t} = \log S^{BS}_{\epsilon,t}$)

i.e. in this regime, the mean-reversion of Y is so fast that $\sigma(Y(\cdot))$ gets averaged to $\bar{\sigma}$ and the stock price behaves effectively like the Black-scholes model with constant volatility $\bar{\sigma}$.

Rate functions ($\delta = \epsilon^2$)

$$I(x'; x, t) = t \sup_{p \in \mathbb{R}} \left\{ p\left(\frac{x - x'}{t}\right) - \bar{H}_0(p) \right\}$$

$\bar{H}_0$

$$\bar{H}_0(p) = \lim_{T \to +\infty} T^{-1} \log E[e^{\frac{1}{2}p^2 \int_0^T \sigma^2(Y_s^p)ds} | Y_0^p = y];$$

Process Y^p is multiplicative ergodic (a strong enough ergodic property that the above limit exists and is independent of $Y_0^p = y$), see Kontoyiannis and Meyn [4] for definition of multiplicative ergodicity.

$$\bar{H}_0(p) = \lim_{T \to +\infty} T^{-1} \log E[e^{\frac{1}{2}p^2 \int_0^T \sigma^2(Y_s^p)ds} | Y_0^p = y]\}$$
$$= \sup_{\mu \in \mathcal{P}(\mathbb{R}_+)} \Big(\frac{|p|^2}{2} \int_{\mathbb{R}_+} \sigma^2 d\mu - J(\mu; p) \Big).$$

Where $J(\cdot; p)$ is the rate function for the LDP of the occupation measures $\{\mu_T(\cdot)\}_{T \geq 0}$:

$\mu_T(A) = \frac{1}{T} \int_0^T \mathbf{1}_{\{Y_s^p \in A\}} ds$ average amount of time Y^p spends in set A.

(See Stroock [5].)

Y^p is a mean-reverting and ergodic process. As $\epsilon \longrightarrow 0$, the distribution of $Y^p_{\epsilon,t}$ approaches its invariant distribution. $J(\cdot; p)$ measures the cost of deviation of $Y^p_{\epsilon,t}$ from its invariant distribution.

Rate functions ($\delta = \epsilon^2$)

$$I(x'; x, t) = t \sup_{p \in \mathbb{R}} \inf_{\mu \in \mathcal{P}(\mathbb{R}_+)} \left\{ p\left(\frac{x - x'}{t}\right) - \frac{|p|^2}{2} \int_{\mathbb{R}_+} \sigma^2 d\mu + J(\mu; p) \right\}$$

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The deviation $\{X_{\epsilon,t} > x'\}$ is caused by a perturbation of $Y_{\epsilon,\cdot}$ to $Y^p_{\epsilon,\cdot}$ and then $Y^p_{\epsilon,\cdot}$ deviates from its invariant distribution.