Small time asymptotics for fast mean-reverting stochastic volatility models

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5 Comments on the rate functions
Let $S_t$ denote stock price at time $t$.

A *European call option* on this stock is a contract that gives its holder the right, but not the obligation to buy a unit of this stock at a certain price, called *strike price*, and at a given time called *maturity time*.
Let $S_t$ denote stock price at time $t$.

A *European call option* on this stock is a contract that gives its holder the right, but not the obligation to buy a unit of this stock at a certain price, called *strike price*, and at a given time called *maturity time*.

Let $K = \text{strike price}$, $T = \text{maturity time}$ and suppose $S_0 < K$ (out-of-the-money).
What is the behavior of

\[ \text{option price} = E[e^{-rT}(S_T - K)^+] \]

as time to maturity \( T \to 0 \)?
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\]

as time to maturity \( T \to 0 \)?

To estimate this quantity, we study

\[
P(S_T > K)
\]

as \( T \to 0 \).

This probability decays exponentially fast to 0. We get a large deviation estimate of the form

\[
\lim_{T \to 0} T \log P(S_T > K) = -I(K)
\]
A simple model for this stock price is the B-S model

\[ dS_t = rS_t dt + \sigma S_t dW_t; \]

\( \sigma > 0 \) is called volatility.

Under the B-S model, the price of a call option with strike price \( K \) and maturity time \( T \) is:

\[ \text{option price} = E[e^{-rT}(S_T - K)^+] \]

is easy to calculate.

However the assumption of constant volatility is unrealistic and we instead work with a more sophisticated model.
Stochastic volatility model for stock price

Let $S_t$ denote stock price.

\begin{align}
    dS_t &= rS_t dt + S_t \sigma(Y_t) dW^{(1)}_t, \\
    dY_t &= \frac{1}{\delta} (m - Y_t) dt + \frac{\nu}{\sqrt{\delta}} Y_t^\beta dW^{(2)}_t.
\end{align}

where $m \in \mathbb{R}$, $r, \nu > 0$, $W^{(1)}$ and $W^{(2)}$ are standard Brownian motions with $\langle W^{(1)}, W^{(2)} \rangle_t = \rho t$, with $|\rho| < 1$ constant.

- The process $(Y_t)$ is a fast mean-reverting process with rate of mean reversion $1/\delta$ ($\delta > 0$).
We assume that

Assumption

1. $\beta \in \{0\} \cup \left[\frac{1}{2}, 1\right]$;

2. *in the case of $\beta = 1/2$, we require $m > \nu^2/2$ and $Y_0 > 0$ a.s., in the case of $1/2 < \beta < 1$, we require $m > 0$ and $Y_0 > 0$ a.s.;*

3. $\sigma(y) \in C(\mathbb{R}; \mathbb{R}_+)$ satisfies

$$\sigma(y) \leq C (1 + |y|^\sigma),$$

*for some constants $C > 0$ and $\sigma$ with $0 \leq \sigma < 1 - \beta$. 
Examples of $Y$ process

- **Ornstein-Uhlenbeck process** (take $\beta = 0$)
  \[ dY_t = \frac{1}{\delta} (m - Y_t) dt + \frac{\nu}{\sqrt{\delta}} dW_t^{(2)}. \]

- **CIR process** (take $\beta = 1/2$)
  \[ dY_t = \frac{1}{\delta} (m - Y_t) dt + \frac{\nu}{\sqrt{\delta}} \sqrt{Y_t} dW_t^{(2)}. \]
Rescaling time

Let $X_t = \log S_t$. Rescale time $t \mapsto \epsilon t$.

\[ dX_{\epsilon, t} = \epsilon \left( r - \frac{1}{2} \sigma^2(Y_{\epsilon, t}) \right) dt + \sqrt{\epsilon} \sigma(Y_{\epsilon, t}) dW_t^{(1)} \]  

(2a)

\[ dY_{\epsilon, t} = \frac{\epsilon}{\delta} (m - Y_{\epsilon, t}) dt + \nu \sqrt{\frac{\epsilon}{\delta}} Y_{\epsilon, t}^{\beta} dW_t^{(2)}. \]  

(2b)

Our mean-reversion time $\delta$ is $\epsilon$-dependent.
Rescaling time

Let \( X_t = \log S_t \). Rescale time \( t \mapsto \epsilon t \).

\[
\begin{align*}
    dX_{\epsilon, t} &= \epsilon \left( r - \frac{1}{2} \sigma^2(Y_{\epsilon, t}) \right) dt + \sqrt{\epsilon} \sigma(Y_{\epsilon, t}) dW_t^{(1)} \\
    dY_{\epsilon, t} &= \frac{\epsilon}{\delta} (m - Y_{\epsilon, t}) dt + \nu \sqrt{\frac{\epsilon}{\delta}} Y_{\epsilon, t}^{\beta} dW_t^{(2)}.
\end{align*}
\]

(2a)

(2b)

Our mean-reversion time \( \delta \) is \( \epsilon \)-dependent.

Consider 2 regimes:

- \( \delta = \epsilon^4 \) (ultra-fast regime)
- \( \delta = \epsilon^2 \) (fast regime)
Rescaling time

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(2b)

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Consider 2 regimes:

- $\delta = \epsilon^4$ (ultra-fast regime)
- $\tilde{\delta} = \epsilon^2$ (fast regime)

As $\epsilon \to 0$, we look at small time asymptotics of $X$ process but large time asymptotics of the $Y$ process. $Y$ is mean-reverting and ergodic and approaches its invariant distribution in large time. The effect of $Y$ gets averaged!
Let \( X_{\epsilon,0} = x \). We prove the following large deviation estimates of the probabilities of \( \{X_{\epsilon,t} > x'\} \) when \( x' > x \).

**THEOREM**

\[
\lim_{\epsilon \to 0} \epsilon \log P(X_{\epsilon,t} > x') = -I(x'; x, t)
\]

with rate functions \( I(x'; x, t) \) as follows.
Rate function ($\delta = \epsilon^4$ case)

When $\delta = \epsilon^4$,

$$I(x'; x, t) = \frac{|x - x'|^2}{2\bar{\sigma}^2 t},$$

where $\bar{\sigma}^2$ is the average of the volatility function $\sigma(y)$ with respect to the invariant distribution of $Y$. 
Rate function (\(\delta = \epsilon^2\) case)

When \(\delta = \epsilon^2\),

\[
I(x'; x, t) = t \sup_{p \in \mathbb{R}} \left\{ p \left( \frac{x - x'}{t} \right) - \bar{H}_0(p) \right\}
\]

where

\[
\bar{H}_0(p) = \lim_{T \to +\infty} T^{-1} \log E\left[ e^{\frac{1}{2} p^2 \int_0^T \sigma^2(Y^p_s) ds} \bigg| Y^p_0 = y \right].
\]

\(Y^p\) is the process with the perturbed \(Y\) process with generator \(B^p\)

\[
B^p g(y) = B g(y) + \rho \sigma \nu y^\beta p \partial_y g(y), \tag{3}
\]

where \(B\) is the generator of the \(Y\) process.
Problem

Two aspects to this problem:

It is a Large Deviation problem coupled with a homogenization problem.
Key Steps in Proof

- Prove convergence of the following functionals

\[ u^h_\epsilon(t, x, y) := \epsilon \log E[e^{\epsilon^{-1}h(X_\epsilon, t)} | X_{\epsilon,0} = x, Y_{\epsilon,0} = y], \quad h \in C_b(\mathbb{R}) \]

to \( u^h_0(t, x) \).
Key Steps in Proof

- Prove convergence of the following functionals

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to \( u_0^h(t, x) \).

- Prove exponential tightness of \( \{X_\epsilon, t\}_{\epsilon > 0} \), i.e. for any \( \alpha > 0 \) there exists a compact set \( K_\alpha \subset \mathbb{R} \) such that

\[ \lim_{\epsilon \to 0} \epsilon \log P(X_\epsilon, t \notin K_\alpha) \leq -\alpha. \]
Key Steps in Proof

- Prove convergence of the following functionals

\[ u^h_\varepsilon(t, x, y) := \varepsilon \log E[e^{-h(X_{\varepsilon, t})} | X_{\varepsilon, 0} = x, Y_{\varepsilon, 0} = y], \quad h \in C_b(\mathbb{R}) \]

[to \( u^h_0(t, x) \)].

- Prove exponential tightness of \( \{X_{\varepsilon, t}\}_{\varepsilon > 0} \), i.e. for any \( \alpha > 0 \) there exists a compact set \( K_\alpha \subset \mathbb{R} \) such that

\[ \lim_{\varepsilon \to 0} \varepsilon \log P(X_{\varepsilon, t} \notin K_\alpha) \leq -\alpha. \]

Then, by Bryc’s inverse Varadhan lemma, \( \{X_{\varepsilon, t}\}_{\varepsilon > 0} \) satisfies a LDP with

\[ I(x'; x, t) := \sup_{h \in C_b(\mathbb{R})} \{ h(x') - u^h_0(t, x) \}. \]
Convergence of $u_\epsilon$

Fix $h \in C_b(\mathbb{R})$. How do we prove

$$u_\epsilon(t, x, y) := \epsilon \log E[e^{\epsilon^{-1}h(X_\epsilon,t)}|X_{\epsilon,0} = x, Y_{\epsilon,0} = y]$$

converges?
Convergence of $u_\epsilon$

Possible ways:

- Compute $u_\epsilon$ and take limit.
- Use PDE (viscosity solution) approach.
- Operator-theoretic approach (See Feng and Kurtz [2]):

$$u^h_\epsilon = S_\epsilon(t)h$$

is a nonlinear contraction semigroup.

Method: Let $H_\epsilon$ denote nonlinear generator of $S_\epsilon(t)$. Prove $H_\epsilon \to H$. Invoke Crandall-Liggett generation theorem to get the limit semigroup $S(t)$ corresponding to $H$.

- A variational representation for $S_\epsilon(t)h$ can be obtained. $S_\epsilon(t)$ can be interpreted as a Nisio semigroup.
$u_\epsilon$ satisfies the following nonlinear pde:

\[ \partial_t u = H_\epsilon u, \quad \text{in } (0, T] \times \mathbb{R} \times E_0; \quad (4a) \]

\[ u(0, x, y) = h(x), \quad (x, y) \in \mathbb{R} \times E_0. \quad (4b) \]

$E_0$ is the state space of $Y$. 
Let $A_\epsilon$ denote the generator of the markov process $(X_\epsilon, Y_\epsilon)$. Then

$$H_\epsilon u(t, x, y) = \epsilon e^{-\epsilon^{-1}u} A_\epsilon e^{\epsilon^{-1}u}(t, x, y)$$

$$= \epsilon \left( (r - \frac{1}{2} \sigma^2) \partial_x u + \frac{1}{2} \sigma^2 \partial_{xx} u \right) + \frac{1}{2} |\sigma \partial_x u|^2$$

$$+ \frac{\epsilon^2}{\delta} e^{-\epsilon^{-1}u} B e^{\epsilon^{-1}u} + \rho \sigma(y) \nu y^\beta \left( \frac{\epsilon}{\sqrt{\delta}} \partial_{xy}^2 u + \frac{1}{\sqrt{\delta}} \partial_x u \partial_y u \right)$$

$$= \epsilon B u + \delta^{-1} \frac{1}{2} |\nu y^\beta \partial_y u|^2.$$  

(5)
We prove that as $\epsilon \to 0$, $u_\epsilon \to u_0$ where $u_0$ satisfies the HJB equation

$$\partial_t u_0 = \frac{1}{2} |\bar{\sigma} \partial_x u_0(x)|^2;$$

$$u_0(0, x) = h(x),$$

where $\bar{\sigma}^2$ is the average of $\sigma^2(\cdot)$ with respect to the invariant distribution of $Y$. 

$u_0$ for $\delta = \epsilon^4$ case
$u_0$ for $\delta = \epsilon^2$ case

We prove that as $\epsilon \to 0$, $u_\epsilon \to u_0$ where $u_0$ satisfies the HJB equation

$$
\partial_t u_0 = \bar{H}_0(\partial_x u_0)
$$

$$
u_0(0, x) = h(x),
$$

where

$$
\bar{H}_0(p) = \lim_{T \to +\infty} T^{-1} \log E[e^{\frac{1}{2}p^2 \int_0^T \sigma^2(Y^p_s)ds}|Y_0^p = y].
$$
Rigorous proof of convergence of $u_\epsilon$

We use viscosity solution techniques adapted from Feng and Kurtz [2].

**Difficulties:**

- In what sense do the operators $H_\epsilon$ converge? How do we identify the limit operator?
- We are averaging over a non-compact space!
- We need to carefully choose suitable perturbed test functions in our proof.
- The perturbed test functions $f_\epsilon(t, x, y)$ and $H_\epsilon f_\epsilon(t, x, y)$ should have compact level sets.
Rigorous proof of convergence of $u_\epsilon$

We prove conditions for convergence of $u_\epsilon$, solutions (in the viscosity solution sense) of 

$$\partial_t u = H_\epsilon u,$$

to a sub-solution of

$$\partial_t u(t, x) \leq \inf_{\alpha \in \Lambda} H_0(x, \nabla u(t, x), D^2 u(t, x); \alpha), \quad (8)$$

and a super-solution of

$$\partial_t u(t, x) \geq \sup_{\alpha \in \Lambda} H_1(x, \nabla u(t, x), D^2 u(t, x); \alpha). \quad (9)$$

The method used is a generalization of Barles-Perthame’s half-relaxed limit arguments first introduced in single scale, compact space setting (see Fleming and Soner [3]).
Consider an out-of-the-money European call option i.e.

\[ S_0 < K \]

or

\[ x = \log S_0 < \log K. \]

**Lemma**

\[
\lim_{\epsilon \to 0^+} \epsilon \log E[e^{-r\epsilon t}(S_{\epsilon,t} - K)^+] = -I(\log K; x, t),
\]

where \( I \) is the rate function for LDP of \( \{X_{\epsilon,t}\}_{\epsilon>0} \).
Asymptotics of option price

Proof.

\[
\lim_{\epsilon \to 0^+} \epsilon \log E[e^{-r\epsilon t}(S_\epsilon, t - K)^+] = \lim_{\epsilon \to 0^+} \epsilon \log \int_{\infty}^{\infty} P(S_\epsilon, t > z)dz
\]

\[
= \lim_{\epsilon \to 0^+} \epsilon \log \int_{K}^{\infty} P(X_\epsilon, t > \log z)dz
\]

\[
\approx \lim_{\epsilon \to 0^+} \epsilon \log \int_{K}^{\infty} \exp \left\{ - \frac{I(\log z; x, t)}{\epsilon} \right\} dz
\]

\[
= - \inf_{z \in (K, \infty)} I(\log z; x, t)
\]

(by Laplace principle)

\[
= -I(\log K; x, t)
\]

as \( I \) is a continuous, increasing function in the interval \((x, \infty)\).
Details of this work can be found on http://arxiv.org/abs/1009.2782.

Our paper titled “Small time asymptotics for fast mean-reverting stochastic volatility models” has been accepted (pending minor revisions) in Annals of Applied Probability.
References


Thank You!
Black-Scholes model

A simple model for this stock price is the B-S model

\[ dS_t = rS_t dt + \sigma S_t dW_t; \]

\( \sigma > 0 \) is called volatility. 

Price of a call option with strike price \( K \) and maturity time \( T \) under this model can be easily calculated:

\[
E^{BS} \left[ e^{-rT} (S_T - K)^+ \right] = S_0 \Phi \left( \frac{\log(S_0/K) + rT + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \right) - Ke^{-rT} \Phi \left( \frac{\log(S_0/K) + rT - \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \right)
\]

(Black-Scholes Formula)
**Implied Volatility**

*Implied Volatility*, $\Sigma(T, K)$, is the volatility parameter value to be inputed in the Black-Scholes model to match a call option price.

Implied volatility for Black-Scholes model is a constant for all $T$ and $K$.

However, implied volatilities of market prices are *not constant* and vary with $T$ and $K$. Keeping $T$ fixed, the graph of implied volatilities of market prices as a function of $K$ is approximately U-shaped.
Implied volatility

Let $\sigma_\varepsilon$ denote the implied volatility corresponding to strike price $K$ of option price given by our stochastic volatility model. Then $\sigma_\varepsilon$ is obtained by solving:

$$e^{r\varepsilon t} S_0 \Phi \left( \frac{x - \log K + r \varepsilon t + \frac{1}{2} \sigma_\varepsilon^2 \varepsilon t}{\sigma_\varepsilon \sqrt{\varepsilon t}} \right) - K \Phi \left( \frac{x - \log K + r \varepsilon t - \frac{1}{2} \sigma_\varepsilon^2 \varepsilon t}{\sigma_\varepsilon \sqrt{\varepsilon t}} \right) = E \left[ e^{-r\varepsilon t} (S_{\varepsilon,t} - K)^+ \right] \approx e^{-\frac{I(\log K;x,t)}{\varepsilon}}$$
Taking \( \lim_{\epsilon \to 0} \epsilon \log \) on both sides, we get

\[
\lim_{\epsilon \to 0^+} \frac{\sigma^2}{\epsilon} = \frac{(\log K - x)^2}{2I(\log K; x, t)t}.
\]
Taking $\lim_{\varepsilon \to 0} \varepsilon \log$ on both sides, we get

$$\lim_{\varepsilon \to 0^+} \sigma_\varepsilon^2 = \frac{(\log K - x)^2}{2I(\log K; x, t)t}.$$

In the regime $\delta = \varepsilon^4$, we get $\lim_{\varepsilon \to 0^+} \sigma_\varepsilon^2 = \bar{\sigma}^2$.

In the regime $\delta = \varepsilon^2$, we need to first compute the quantity $\bar{H}_0$ defined as the limit of a log moment. This can be computed for the Heston model i.e. when $\sigma(y) = \sqrt{y}$ and $\beta = 1/2$. 
A LDP and applications to option pricing and implied volatility for the Heston Model are in Feng, Forde and Fouque [1].

Here \( x = \log(K/S_0) \). Take \( \sigma(y) = \sqrt{y} \) and the parameters are \( t = 1, \beta = 1/2, m = .04, \nu = 1.74 \) and \( \rho = - .4 \) (dashed blue), \( \rho = 0 \) (solid black), \( \rho = + .4 \) (dotted red).
Rate functions \((\delta = \epsilon^4)\)

\[
I(x'; x, t) = \frac{|x' - x|^2}{2\bar{\sigma}^2 t}
\]

is the rate function for \(\{Z_{\epsilon, t}\}_{\epsilon > 0}\) where \(Z_{\epsilon, \cdot}\) satisfies \(Z_{\epsilon, 0} = x\) and

\[
Z_{\epsilon, t} = x + \epsilon \int_0^t \left( r - \frac{1}{2} \bar{\sigma}^2 \right) ds + \sqrt{\epsilon} \int_0^t \bar{\sigma} dW_s
\]

\((Z_{\epsilon, t} = \log S_{\epsilon, t}^{BS})\)
Rate functions ($\delta = \epsilon^4$)

$$I(x'; x, t) = \frac{|x' - x|^2}{2\bar{\sigma}^2 t}$$

is the rate function for $\{Z_{\epsilon, t}\}_{\epsilon > 0}$ where $Z_{\epsilon, t}$ satisfies $Z_{\epsilon, 0} = x$ and

$$Z_{\epsilon, t} = x + \epsilon \int_0^t \left( r - \frac{1}{2} \bar{\sigma}^2 \right) ds + \sqrt{\epsilon} \int_0^t \bar{\sigma} dW_s$$

($Z_{\epsilon, t} = \log S^{BS}_{\epsilon, t}$)

i.e. in this regime, the mean-reversion of $Y$ is so fast that $\sigma(Y(\cdot))$ gets averaged to $\bar{\sigma}$ and the stock price behaves effectively like the Black-scholes model with constant volatility $\bar{\sigma}$.
Rate functions \((\delta = \epsilon^2)\)

\[
l(x'; x, t) = t \sup_{p \in \mathbb{R}} \left\{ p \left( \frac{x - x'}{t} \right) - \bar{H}_0(p) \right\}
\]
\[ \bar{H}_0(p) = \lim_{T \to +\infty} T^{-1} \log E\left[ e^{\frac{1}{2}p^2 \int_0^T \sigma_s^2(Y^p_s)ds} \bigg| Y^p_0 = y \right]; \]

Process $Y^p$ is multiplicative ergodic (a strong enough ergodic property that the above limit exists and is independent of $Y^p_0 = y$), see Kontoyiannis and Meyn [4] for definition of multiplicative ergodicity.
\[ \bar{H}_0(p) = \lim_{T \to +\infty} T^{-1} \log E[e^{\frac{1}{2}p^2 \int_0^T \sigma^2(Y^p_s)ds} | Y^p_0 = y] \]

\[ = \sup_{\mu \in \mathcal{P}(\mathbb{R}_+)} \left( \frac{|p|^2}{2} \int_{\mathbb{R}_+} \sigma^2 d\mu - J(\mu; p) \right). \]

Where \( J(\cdot; p) \) is the rate function for the LDP of the occupation measures \( \{\mu_T(\cdot)\}_{T \geq 0} \):

\[ \mu_T(A) = \frac{1}{T} \int_0^T 1_{\{Y^p_s \in A\}} ds \quad \text{average amount of time } Y^p \text{ spends in set } A. \]

(See Stroock [5].)
$Y^p$ is a mean-reverting and ergodic process. As $\epsilon \to 0$, the distribution of $Y_{\epsilon,t}^p$ approaches its invariant distribution. $J(\cdot; p)$ measures the cost of deviation of $Y_{\epsilon,t}^p$ from its invariant distribution.
Rate functions \((\delta = \epsilon^2)\)

\[
I(x'; x, t) = t \sup_{p \in \mathbb{R}} \inf_{\mu \in \mathcal{P}(\mathbb{R}^+)} \left\{ p\left(\frac{x - x'}{t}\right) - \frac{|p|^2}{2} \int_{\mathbb{R}^+} \sigma^2 d\mu + J(\mu; p) \right\}
\]
Rate functions \((\delta = \epsilon^2)\)

\[
l(x'; x, t) = t \sup_{p \in \mathbb{R}} \inf_{\mu \in \mathcal{P}(\mathbb{R}_+)} \left\{ p \left( \frac{x - x'}{t} \right) - \frac{|p|^2}{2} \int_{\mathbb{R}_+} \sigma^2 d\mu + J(\mu; p) \right\}
\]

The deviation \(\{X_{\epsilon,t} > x'\}\) is caused by a perturbation of \(Y_{\epsilon, \cdot}\) to \(Y_{\epsilon, \cdot}^p\), and then \(Y_{\epsilon, \cdot}^p\) deviates from its invariant distribution.