The Chaotic Character of the Stochastic Heat Equation

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- Intermittency
- The Stochastic Heat Equation
- Blowup of the solution

Intermittency-Example

$\xi_j, j = 1, 2, \cdots, 10$ i.i.d. random variables

Taking values 0 and 2 with probability 1/2 each

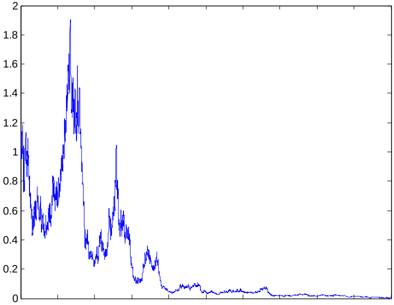
$$\eta = \Pi_{j=1}^{10} \xi_j$$

- $\eta = 0$ with probability $1 - \frac{1}{2^{10}}$
- $\eta = 2^{10}$ with probability $\frac{1}{2^{10}}$.
- The moments $E\eta^p = 2^{10(p-1)}$

Intermittency-Exponential Martingale

$$dX_t = X_t dB_t \quad , \qquad X_0 = 1$$

- The solution is $X_t = \exp\left(B_t - \frac{t}{2}\right) \approx \Pi \exp\left(B_{t_i+\Delta t_t} - B_{t_i} - \frac{\Delta t_i}{2}\right)$.
- $X_t \to 0$ as $t \to \infty$.
- $E\left(X_t^p\right) = \exp\left(\frac{p(p-1)}{2} t\right)$



Intermittency-Stochastic heat equation on lattice

$$\frac{\partial u}{\partial t} = \kappa \Delta u + Wu, \qquad u(0,\cdot) = 1$$

W is a Gaussian noise that is brownian in time and with "nice" homogeneous spatial correlations.

$$u(t,x) = E_Y\left[\exp\left(\int_0^t W(ds, Y_t - Y_s + x)\right)\right]$$

Y: Continuous time random walk with jump rate κ.

- $\gamma_p = \lim_{t\to\infty} \frac{\log E[|u(t,x)|^p]}{t}$ (Moment Lyapunov Exponent)
- If κ is small, $\gamma_1 < \frac{\gamma_2}{2} < \frac{\gamma_3}{3} < \cdots$ (Mathematical Intermittency)
- Implies the existence of rare and intense peaks in the space-time profile of $u(t,x)$

Some interesting results

- When the Gaussian noise W is independent Brownian motions, then $\lim_{t\to\infty} \frac{\log u(t,x)}{t} \approx \frac{C}{\log \frac{1}{\kappa}}$ **[Cranston, Mountford, Shiga]**
- For $\frac{\partial u}{\partial t} u(t,z) = \Delta u(t,z) + \xi(z) u(t,z),\ u(0,\cdot) = \mathbf{1}_0$ and ξ is i.i.d. with tails heavier than double-exponential, the radius of these "intermittent islands" are bounded. **[Gärtner, König, Molchanov]**
- If ξ has i.i.d Pareto distribution $P(\xi(z) \leq x) = 1 - x^{-\alpha},\ x \geq 1$ for $\alpha > d$, then almost all the mass is concentrated on **two** random points. **[König et al.]**

Intermittency-The Universe

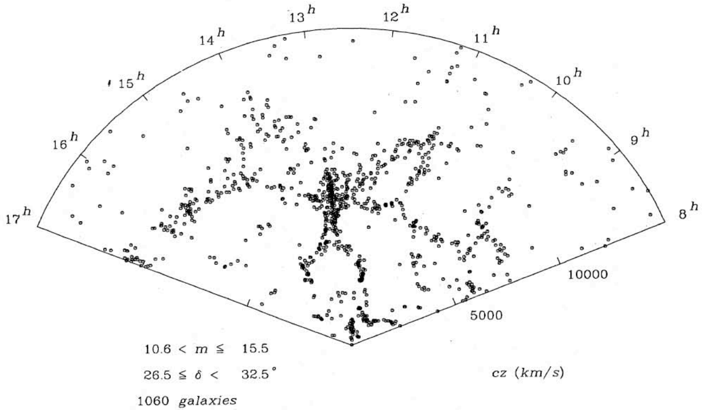
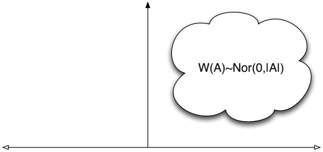


FIG. 1. The distribution of galaxies in a thin slice with $8^h \leq \alpha \leq 17^h$ and $26°.5 \leq \delta \leq 32°.5$, where α (right ascension) and δ (declination) are spherical coordinates (de Lapparent *et al.*, 1986). The positions of 1060 galaxies with $m_B \leq 15.5$ and $v \leq 15\,000$ km s^{-1} are indicated. The scale shows the velocities of the galaxies, and their distances can be estimated assuming that the velocity and the distance of a galaxy are related according to Hubble's law, $v = H_0 r$ ($H_0 = 50 h_{50}$ km s^{-1}Mpc^{-1}).

White Noise

White noise $\dot{W}$ on $\mathbb{R}_+ \times \mathbb{R}$ is a Gaussian process indexed by Borel subsets of $\mathbb{R}_+ \times \mathbb{R}$.



- For $A \subset \mathbb{R}_+ \times \mathbb{R}$, $\dot{W}(A) \sim N(0, |A|)$.
- For A, $B \subset \mathbb{R}_+ \times \mathbb{R}$, $E\left[\dot{W}(A)\dot{W}(B)\right] = \left|A \cap B\right|$.
- Can define $\int h \, \dot{W}(dsdx)$ for $h \in L^2\left(\mathbb{R}_+ \times \mathbb{R}\right)$.
- Can also integrate "predictable functions" with respect to white noise.

The Stochastic Heat Equation

(SHE) $\quad u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2} \frac{\partial^2}{\partial x^2} u + \sigma(u) \dot{W}(t,x), \qquad u(0,\cdot) = u_0(\cdot) \text{ bounded nonnegative}$$

$\dot{W}(t,x)$ is a 2 parameter white noise and $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz.

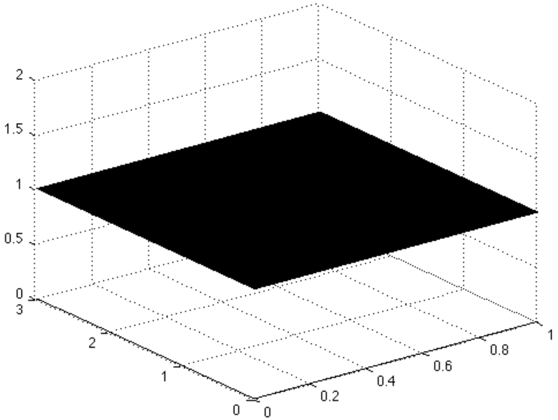
The (SHE) has an a.s. unique solution (that is bounded in L^2) given by

$$u(t,x) = \int_{\mathbb{R}} p_t(y-x) u_0(y) + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x) \sigma\left(u(s,y)\right) \dot{W}(dy, ds)$$

where $p_t(x) = \frac{1}{\sqrt{2\kappa\pi t}} \exp\left(-\frac{x^2}{2\kappa t}\right)$

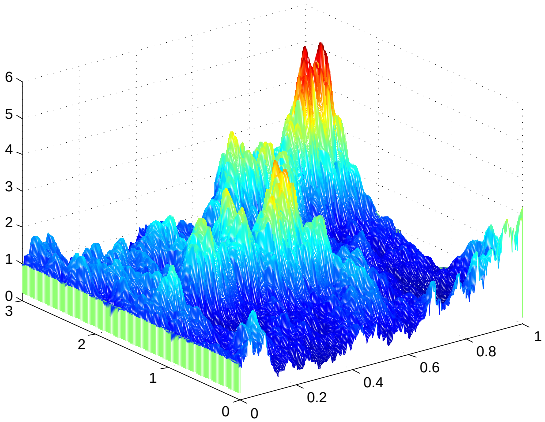
- The SHE does not have a solution in higher spatial dimensions
- Not known if a solution exists if σ is not Lipschitz.

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u, \qquad u(0,\cdot) = 1$$



Stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\dot{W}, \qquad u(0,\cdot) = 1$$



Intermittency for SHE

Theorem (Foondun, Khoshnevisan)

If $|\sigma(u)| \geq C|u|$ and $\inf_x u_0(x) > 0$, then the solution to the SHE is intermittent.
If $\sigma(u)$ is bounded, intermittency does not occur.

- Parabolic Anderson Model : $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\dot{W}$
- $\log u$ is a proposed solution to the KPZ equation
- Turbulence, chemical kinetics, branching processes in random environment

Theorem (Bertini, Giacomin)

For the PAM and $u_0(x) = e^{B_x}$ (where B_x is a two sided brownian motion) and $\phi \in C_0^\infty(\mathbb{R})$

$$\lim_{t\to\infty} \frac{(\log u(t,\cdot), \phi)}{t} = -\frac{1}{24}(1,\phi) \ \text{ in } L^2$$

- If $\phi = \delta_0$ then $\frac{\log u(t,0)}{t} \to -\frac{1}{24}$ in probability !
- Believed to be true for other initial conditions

Blowup of the solution to SHE

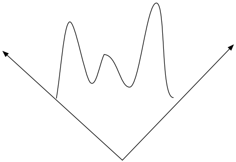
We are interested in the behavior of $u_t^*(R) = \sup_{|x|\le R} u(t,x)$.

- In the case of the heat equation, $u_t^*(R)$ is bounded by $\sup_x u_0(x)$.
- For the SHE, does $u_t^*(R) \to \infty$?

Theorem (Foondun, Khoshnevisan)

If $|\sigma(u)| \ge C|u|$ and $u_0 \not\equiv 0$ is compact and Holder continuous of order $\ge 1/2$, then

$$0 < \limsup_{t\to\infty} \frac{1}{t} E\left[\sup_x |u(t,x)|^2\right] < \infty$$



- The highest peaks occur within $[-Ct, Ct]$ **[Conus, Khoshnevisan]**

Blowup of the solution to SHE

- Assume $\inf_x u_0(x) > 0$. **Is this necessary?**

Theorem (Mueller's comparison theorem)

Suppose $u^{(1)}$ and $u^{(2)}$ are solutions to $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sigma(u)\dot{W}$ with $u^{(1)}(0,\cdot) \leq u^{(2)}(0,\cdot)$. Then

$$u^{(1)}(t,\cdot) \leq u^{(2)}(t,\cdot)$$

- For blowup, need $\sigma(x) \neq 0$ for $x > 0$. **Is this sufficient?**

Blowup of the solution to SHE

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2}\Delta u + \sigma(u)\dot{W}$$

Theorem (Conus, Joseph, Khoshnevisan)

- *If* $\inf_x \sigma(x) \geq \epsilon_0$, *then*

$$\liminf_{R\to\infty} \frac{u_t^*(R)}{(\log R)^{\frac{1}{6}}} > 0 \ a.s.$$

- *If* $\epsilon_1 \leq \sigma(x) \leq \epsilon_2$ *for all* x, *then*

$$u_t^*(R) \asymp \frac{(\log R)^{1/2}}{\kappa^{1/4}} \ \ a.s.$$

- *For the Parabolic Anderson Model with* $\sigma(x) = cx$,

$$\log u_t^*(R) \asymp \frac{(\log R)^{2/3}}{\kappa^{1/3}} \ \ a.s.$$

Colored noise case

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sigma(u)\dot{F}$$

F is spatially homogeneous Gaussian noise which is Brownian in time and with spatial correlation function $f = h * \tilde{h}$, $h \in L^2(\mathbb{R})$

Theorem (Conus, Joseph, Khoshnevisan)

- *If* $\inf_x \sigma(x) \geq \epsilon_0$, *then*

$$\liminf_{R\to\infty} \frac{u_t^*(R)}{(\log R)^{\frac{1}{4}}} > 0 \ a.s.$$

- *If* $\epsilon_1 \leq \sigma(x) \leq \epsilon_2$ *for all* x, *then*

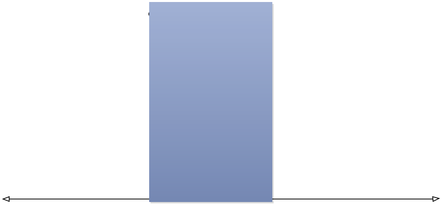
$$u_t^*(R) \asymp (\log R)^{1/2} \quad a.s.$$

- *For the Parabolic Anderson Model with* $\sigma(x) = cx$,

$$\log u_t^*(R) \asymp (\log R)^{1/2} \quad a.s.$$

Creating independence

$$u(t,x) = p_t * u_0(x) + \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\sigma\left(u(s,y)\right) W(dyds)$$



Split into blocks of size $\beta\sqrt{t}$

$$U^{(\beta)}(t,x) = p_t * u_0(x) + \int_{(0,t)\times\mathcal{I}_t^{(\beta)}(x)} p_{t-s}(y-x)\sigma\left(U^{(\beta)}(s,y)\right) W(dyds)$$

$$E\left(\left|u(t,x) - U^{(\beta)}(t,x)\right|^k\right) \leq e^{Ck^3}\beta^{-k/4}$$

Upper bounds on moments

$$||u||_{k,\beta} = \sup_{t\geq 0} e^{-\beta t}\|u(t,0)\|_k$$

Burkholder's inequality

$$\|u(t,x)\|_k \leq C + C_k\sqrt{\int_0^t\int_{\mathbb{R}} p_{t-s}(y-x)^2\Big(\sigma(0)^2 + Lip_\sigma\|u(s,y)\|_k^2\Big)dyds}$$

Multiply both sides by $e^{-\beta t}$ and take sup over t

$$\|u\|_{k,\beta} \leq C + \frac{\sqrt{k}}{(4\kappa\beta)^{1/4}}\left(|\sigma(0)| + Lip_\sigma\|u\|_{k,\beta}\right)$$

Choose β in terms of k so that $\frac{\sqrt{k}}{(4\kappa\beta)^{1/4}} < 1$

- $E\left[u(t,x)^k\right] \leq e^{Ck^3}$

Thank you!