SRW $\mathbb{Z}^d$: CLT & $d=2$ rec, $d\geq 3$ trans.

Balanced RWRE: $\omega=(\omega_x)_{x\in\mathbb{Z}^d}$ environment ; $\omega_x=(\omega_x(e))_{|e|=1}$

P^* measure on ω $\qquad \sum_e \omega_x(e)=1$

RWRE $\{X_n\}_{n\geq 0}$ $\quad P_\omega\{X_{n+1}=x+e \mid X_n=x\}=\omega_x(e)$

Balanced environment: $P^*\{\omega_x(e)=\omega_x(-e)\}=1$ $\forall x, \forall e$; P^* iid

Th. (Lawler '83) $\left.\begin{array}{l} P^*(\omega_0(e)\geq\varepsilon)=1 \ \forall e \\ P^* \text{ ergodic}\end{array}\right\} \Rightarrow$ CLT for P^*-almost all choice of ω

Th. (G-Zeitouni) P^* iid, $P^*(\omega_0(e)>0)=1$ $\forall e \Rightarrow$ same

Cor. $d=2$ rec. (Kesten: CLT, $d=2 \Rightarrow$ rec.)

Also can prove $d\geq 3$ transient

Pf of CLT Th. X_n is a martingale under P_ω.

Need to find an ergodic measure $\sim P^*$ for process viewed from the particle $\theta^{X_n}\omega$

Idea: periodize $\rightsquigarrow$ Markov chain on a finite state space
$\rightsquigarrow$ invariant measure Q_N

"$Q_N \Rightarrow Q$" invariant $\quad$ Now need $Q\sim P^*$. E.g. unif integrability of $\frac{dQ_N}{dP^*}$ $\leftarrow \tilde{\Phi}_N$

$\tilde{\Phi}_N h = \tilde{\Phi}_N \frac{1}{\tau}\sum_{i=1}^{\tau} P_\omega^{(n)} h$

τ: eixt time to exit Box size N

$\left|\sum_{x\in B_N} \tilde{\Phi}_N^\alpha\right|^{\frac{1}{\alpha}} \sim \tilde{\Phi}_N \frac{1}{N^2} \underbrace{E\sum_{n=1}^{\tau} h(X_n)}_{u}$

Alex-Baklman-Pucci $\Rightarrow$ control of $u \Rightarrow \leq \|\tilde{\Phi}_N \varepsilon(\omega)\|$

$\varepsilon(\omega)$: $|\prod \omega(e)|^{\frac{1}{d}}$

Percolation $\rightarrow$ control of ε small (which implies $\tilde{\Phi}_N$ large bad)