Large Deviations for partition functions.

(...and for other polymer related quantities)

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Layout

1. Inroduction and Results.
   - Quadrant directed polymers
   - The Results
   - Boundary *log*-gamma model
   - Burke Property

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   - Quadrant directed polymers
   - The Results
   - Boundary *log*-gamma model
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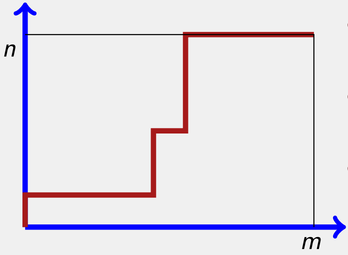
2. The proofs.
   - Decomposition and Burke
   - Estimation
   - Thank you note !
   - Inversion
   - Proofs for unconstrained endpoint model

Directed polymer in space-time random environment



- Nearest neighbor, up-right path $(x(u))$, $u \in \mathbb{N}^2$.
- "Space-time" environment $\{\omega(u) : u \in \mathbb{N}^2\}$.
- $\Pi(m,n) =$ Up-right paths $(x_{m,n})$ from $(1,1)$ to (m,n).
- Quenched probability measure on the paths

$$Q_{m,n}\{x_{m,n}(\cdot)\} = \frac{1}{Z_{m,n}} \exp\left\{\beta \sum_{u \in x_{m,n}(\cdot)} \omega(u)\right\}$$

Inverse temperature $\beta > 0$ (set to be 1 for the majority of the talk) .
The normalizing constant $Z_{m,n}$ is the *partition function*, given by

$$Z_{m,n} = \sum_{x \in \Pi(m,n)} \exp\left\{\beta \sum_{u \in x(\cdot)} \omega(u)\right\}$$

$\mathbb{P}$ is the probability distribution on the environment ω, $\{\omega(u)\}$ i.i.d.

Questions and previous results

Question: Large deviations or Concentration Inequalities for the partition function.

- Concentration Inequalities:
  1. Carmona - Hu: Order n concentration inequality (Gaussian environment)
  2. Comets - Shiga - Yoshida: Order $n^{1/3}$ concentration inequality.
  3. Liu - Watbled: Order n concentration inequality.

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  1. Carmona - Hu: Upper and lower tails normalizations (Gaussian environment)
  2. Ben-Ari: Lower tail large deviation regimes.

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Explicit rate functions for partition functions ?

Existence of Rate functions

Assumptions:

- d-dimensional rectangle (only steps parallel to the positive axes are allowed).
- General i.i.d. weights, so that a $\xi > 0$ that depends on the distributions of the weights ω exists, such that

$$\mathbb{E}\left(e^{\xi|\omega(u)|}\right) < \infty.$$

- $\beta < \xi$.

Theorem

For $t > 0$, $u \in \mathbb{R}^d_+$ and $r \in \mathbb{R}$ there exists a nonnegative function that satisfies

$$J^\beta_u(r) = -\lim_{n\to\infty} n^{-1} \log \mathbb{P}\{\log Z^\beta_{\lfloor nu \rfloor} \geq nr\}.$$

J is convex in the variable (u, r). The rate function is continuous in (u, r) where it is finite.

Log-Gamma Model

- Dimension $1+1$
- I.i.d. weights, with distributions

$$\omega(i,j) \sim \log Y_{i,j}, \quad \text{where } Y_{i,j}^{-1} \sim Gamma(\mu)$$

Gamma density: $\Gamma(\mu)^{-1} x^{\mu-1} e^{-x}$

- The partition function satisfies a law of large numbers:

$$\lim_{n\to\infty} n^{-1} \log Z_{\lfloor ns \rfloor, \lfloor nt \rfloor} = f_\mu(s,t).$$

- $J_{s,t}(r) = -\lim_{n\to\infty} n^{-1} \log \mathbb{P}\{\log Z_{\lfloor ns \rfloor, \lfloor nt \rfloor} \geq nr\}.$

Results - The log-gamma rate function (fixed endpoint)

Definitions:

For $0 < \xi < (\mu + \xi)/2 \leq \theta < \mu$, define

- $h_\xi(\theta) = \log \Gamma(\mu - \theta) - \log \Gamma(\mu - \theta + \xi)$.
- $d_\xi(\theta) = \log \Gamma(\theta - \xi) - \log \Gamma(\theta)$.

Theorem

Let $r \in \mathbb{R}$, $0 \leq s \leq t$. Then

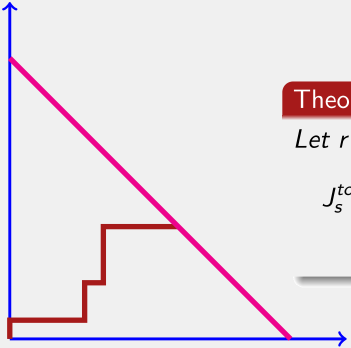
$$J_{s,t}(r) = \sup_{\xi \in [0,\mu)} \sup_{h_\xi((\mu+\xi)/2) \leq v} \left\{ (r, -s) \cdot (\xi, v) - t(d_\xi \circ h_\xi^{-1})(v) \right\},$$

$$= \sup_{\xi \in [0,\mu)} \left\{ r\xi - \inf_{\theta \in [(\mu+\xi)/2, \mu)} \{ s h_\xi(\theta) + t d_\xi(\theta) \} \right\},$$

The function $J_{s,t}(r)$ is strictly positive for $r > r_0 = f_\mu(s, t)$.

Let $s > 0$. The free-endpoint directed polymer model has partition function

$$Z^{tot}_{\lfloor ns \rfloor} = \sum_{x:\lfloor ns \rfloor\text{-paths } x} \exp\Big\{ \sum_{u \in x(\cdot)} \omega(u) \Big\}$$



Theorem

Let $r \in \mathbb{R}$, $s > 0$. Then

$$\begin{aligned} J^{tot}_s(r) &= -\lim_{n\to\infty} n^{-1} \log \mathbb{P}\{\log Z^{tot}_{\lfloor ns \rfloor} \geq nr\} \\ &= J_{s/2,s/2}(r). \end{aligned}$$

Results - Exit point and Path Chain Quenched LDP

Theorem (Exit point LDP)

Let $0 < t < 1$ *and consider* $(t, 1-t) \in \mathbb{R}^2_+$. *For* $n \in \mathbb{N}$ *let* $[n(t, 1-t)] = (\lfloor nt \rfloor, n - \lfloor nt \rfloor)$ *and denote by* x_n *the last point of the polymer chain* $x_{0,n}$. *Then*

$$f_\mu(1/2, 1/2) - f_\mu(t, 1-t) = -\lim_{n\to\infty} n^{-1} \log Q^\omega_n\{x_n = [n(t, 1-t)]\}.$$

This readily leads to the following path large deviations.

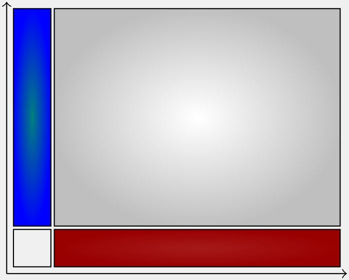
Theorem (Polymer chain LDP)

Let $\gamma(t) : [0,1] \mapsto \mathbb{R}^2_+$ *be a non decreasing Lipschitz-1 curve with* $\gamma(0) = 0$ *and let* $\varepsilon > 0$ *and* $\mathcal{N}_\varepsilon(\gamma)$ *an* ε*-neighborhood of* γ. *Let* $\|\gamma(1)\|_1 = 1$. *Then,*

$$f_\mu(1/2, 1/2) - \int_0^1 f_\mu(\gamma'(t))\, dt = -\lim_{\varepsilon\to 0}\lim_{n\to\infty} n^{-1} \log Q^\omega_n\Big\{x_{0,n} \in n\mathcal{N}_\varepsilon(\gamma)\Big\}.$$

Computational tool: Log-gamma polymer with boundary

Define multiplicative weights $Y_{i,j} = e^{\omega(i,j)}$ independent.



Environment (distribution $\mathbb{P}$)

Horizontal Weights U
$Y_{i,0}^{-1} = U_{i,0}^{-1} \sim Gamma(\theta)$

Vertical Weights V
$Y_{0,j}^{-1} = V_{0,j}^{-1} \sim Gamma(\mu - \theta)$

Bulk Weights Y
$Y_{i,j}^{-1} \sim Gamma(\mu)$

- Gamma(μ) density: $\Gamma(\mu)^{-1} x^{\mu-1} e^{-x}$.

This model allows specific calculations.

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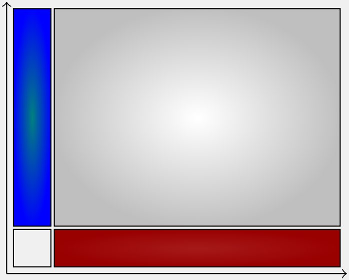

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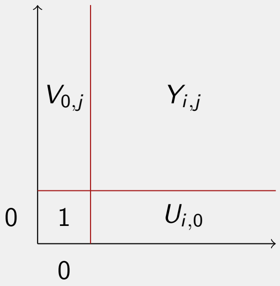
This model allows specific calculations. Reason: Burke property.

Burke Property for the boundary model.



Given initial weights $(i, j \in \mathbb{N})$

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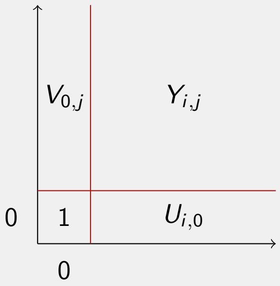

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Compute $Z_{m,n}$ for all $(m, n) \in \mathbb{Z}_+^2$ and then define

$$U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \qquad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}} \qquad X_{m,n} = \left(\frac{Z_{m,n}}{Z_{m+1,n}} + \frac{Z_{m,n}}{Z_{m,n+1}}\right)^{-1}$$

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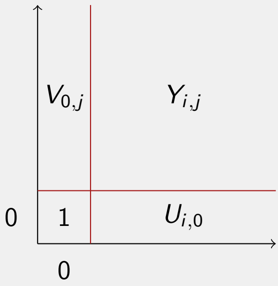

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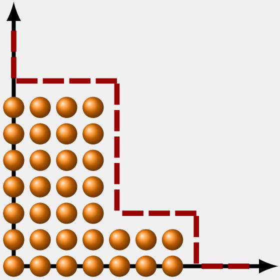
For an undirected edge f: $\quad T_f = \begin{cases} U_x & f = \{x - e_1, x\} \\ V_x & f = \{x - e_2, x\} \end{cases}$

Burke Property

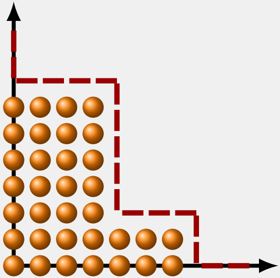


 down-right path (z_k) with edges

$$f_k = \{z_{k-1}, z_k\}, k \in \mathbb{Z}$$

 interior points $\mathcal{I}$ of path (z_k)

Burke Property



down-right path (z_k) with edges

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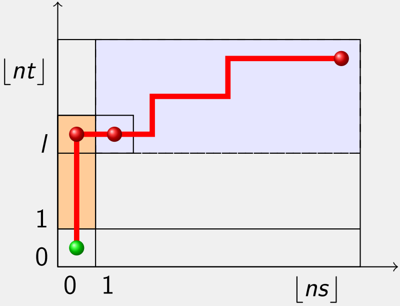
Theorem

Variables $\{\,T_{f_k}, X_z : k \in \mathbb{Z},\ z \in \mathcal{I}\,\}$ *are independent with marginals* $U^{-1} \sim Gamma(\theta)$, $V^{-1} \sim Gamma(\mu - \theta)$, *and* $X^{-1} \sim Gamma(\mu)$.

Idea of Proof - Decomposition

We start by decomposing $Z_{\lfloor ns \rfloor, \lfloor nt \rfloor}$ according to the exit point of the polymer path from the boundary:



The shaded part represents the partition function conditioned on $(0, l)$ being the exit point of the polymer path from the boundary.

$$\left(\prod_{j=1}^{l} V_{0,j} \right) Z^{\square}_{(1,l)}(\lfloor ns \rfloor, \lfloor nt \rfloor)$$

Idea of Proof - Decomposition

$$Z_{\lfloor ns \rfloor, \lfloor nt \rfloor} = \sum_{x.} \prod_{j=1}^{\lfloor ns \rfloor + \lfloor nt \rfloor} Y_{x_j}$$

$$= \sum_{l=1}^{\lfloor nt \rfloor} \left\{ \left(\prod_{j=1}^{l} V_{0,j} \right) Z^{\square}_{(1,l)}(\lfloor ns \rfloor, \lfloor nt \rfloor) \right\} + \sum_{k=1}^{\lfloor ns \rfloor} \left\{ \left(\prod_{i=1}^{k} U_{i,0} \right) Z^{\square}_{(k,1)}(\lfloor ns \rfloor, \lfloor nt \rfloor) \right\}.$$

Consequence of the Burke Property:

$$Z_{\lfloor ns \rfloor, \lfloor nt \rfloor} = \prod_{j=1}^{\lfloor nt \rfloor} V_{0,j} \prod_{i=1}^{\lfloor ns \rfloor} U_{i, \lfloor nt \rfloor}$$

Idea of Proof - Burke Trick

Divide both sides by $\prod_{j=1}^{\lfloor nt \rfloor} V_{0,j}$:

$$\prod_{i=1}^{\lfloor ns \rfloor} U_{i,\lfloor nt \rfloor} = \sum_{l=1}^{\lfloor nt \rfloor} \Big\{ \Big(\prod_{j=l+1}^{\lfloor nt \rfloor} V_{0,j}^{-1} \Big) Z^{\square}_{(1,l)}(\lfloor ns \rfloor, \lfloor nt \rfloor) \Big\}$$

$$+ \sum_{k=1}^{\lfloor ns \rfloor} \Big\{ \Big(\prod_{j=1}^{\lfloor nt \rfloor} V_{0,j}^{-1} \prod_{i=1}^{k} U_{i,0} \Big) Z^{\square}_{(k,1)}(\lfloor ns \rfloor, \lfloor nt \rfloor) \Big\}.$$

$$= \sum_{\substack{k=-\lfloor nt \rfloor \\ k \neq 0}}^{\lfloor ns \rfloor} \eta_k Z^{\square}_{\mathbf{k}}(\lfloor ns \rfloor, \lfloor nt \rfloor).$$

Here we defined

$$\eta_k = \begin{cases} \displaystyle\prod_{j=-k}^{\lfloor nt \rfloor} V_{0,j}^{-1}, & \text{for } -\lfloor nt \rfloor \le k \le -1, \\ \eta_{-1}, & k = 0 \\ \displaystyle\eta_0 \prod_{i=1}^{k} U_{i,0}, & \text{for } 0 < k \le \lfloor ns \rfloor, \end{cases}$$

Idea of Proof - Estimation

$$
\begin{aligned}
R_s(r) &= -\lim_{n\to\infty} n^{-1}\log\mathbb{P}\left\{\log\prod_{i=1}^{\lfloor ns\rfloor} U_{i,\lfloor nt\rfloor} \geq nr\right\}\\
&\sim -n^{-1}\log\mathbb{P}\left\{\log\sum_{\substack{k=-\lfloor nt\rfloor\\ k\neq 0}}^{\lfloor ns\rfloor} \eta_k Z^{\square}_{\mathbf{k}}(\lfloor ns\rfloor,\lfloor nt\rfloor)\geq nr\right\}\\
&\sim -n^{-1}\log\mathbb{P}\{\max_k \log\eta_k Z^{\square}_{\mathbf{k}}(\lfloor ns\rfloor,\lfloor nt\rfloor)\geq nr\}\\
&\sim -n^{-1}\max_k \log\mathbb{P}\{\log\eta_k + \log Z^{\square}_{\mathbf{k}}(\lfloor ns\rfloor,\lfloor nt\rfloor)\geq nr\}\\
&\sim \inf_{-t\leq a\leq s}\inf_{x\in\mathbb{R}}\{\kappa_a(x)+J^a_{s,t}(r-x)\}
\end{aligned}
$$

Thank You...

Questions & Comments ?

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Idea of Proof - Inversion

$$R_s^*(\xi) = \sup_{-t \le a \le s} \left\{ \kappa_a^*(\xi) + (J_{s,t}^a)^*(\xi) \right\}$$

$$F^s(\theta,\xi) = \sup_{-t \le a \le s} \left\{ au(\theta) - (-(J_{s,t}^a)^*(\xi)) \right\}$$

$$F^t(u^{-1}(v),\xi) \overset{\text{magic}}{=} \sup_{0 \le a \le t} \left\{ av - (-(J_{t,t}^a)^*(\xi)) \right\}$$

$$F^t(u^{-1}(v),\xi) = \sup_{0 \le a \le t} \left\{ av - G_\xi(a) \right\}$$

$$F^t(u^{-1}(v),\xi) = G_\xi^*(v)$$

Then,

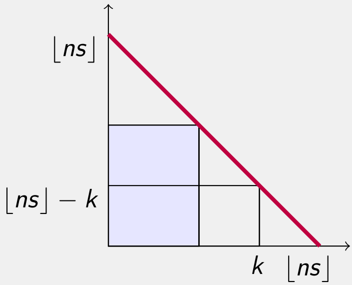
$$\begin{aligned} J_{t,t}^a(r) &= \sup_{\xi \in [0,\mu)} \{ r\xi - (J_{t,t}^a)^*(\xi) \} \\ &= \sup_{\xi \in [0,\mu)} \{ r\xi + G_\xi(a) \} \\ &= \sup_{\xi \in [0,\mu)} \sup_{v \in \mathbb{R}} \{ r\xi + av - G_\xi^*(v) \}. \quad \square \end{aligned}$$

Proof for unconstrained endpoint model

$$\log\left(Z_{\lfloor n\frac{s}{2}\rfloor,\lfloor ns\rfloor-\lfloor n\frac{s}{2}\rfloor}\right) \le \log Z^{tot}_{\lfloor ns\rfloor} \le$$
$$\le \log(ns+1) + \log\left(\max_k Z_{k,\lfloor ns\rfloor-k}\right).$$

After some estimates, this translates to the rate functions:

$$\inf_{0\le a\le s} J_{a,s-a}(r) \le J^{tot}_s(r) \le J_{s/2,s/2}(r).$$



Use convexity of the point-to-point rate functions to get that

$$J_{s/2,s/2}(r) \le \inf_{0\le a\le s} J_{a,s-a}(r).$$