Large deviations in infinite dimensional stochastic analysis

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Goal

- Many Markov processes can be written as solutions of SDE. Hence an approach to **large deviation principle (LDP)** of Markov processes is through study of LDP of SDEs.
  That is assuming that $\{Y_n\}$ satisfies a LDP, we want to study the LDP of $\{X_n\}$, where

$$X_n(t) = X_n(0) + F_n(X_{n-}) \cdot Y_n(t).$$

Definition

Let $(E, \mathcal{E})$ be a regular Hausdorff topological space. A sequence of probability measures $\{\mu_n\}$ on E satisfies a **LDP** with a rate function I if

1. $\liminf_{n\to\infty} n^{-1} \log \mu_n(U) \geq -I(U)$, for every open set $U \in \mathcal{E}$
2. $\limsup_{n\to\infty} n^{-1} \log \mu_n(F) \leq -I(F)$, for every closed set $F \in \mathcal{E}$

where $I(A) = \inf_{x\in A} I(x)$.

Examples

- Freidlin-Wentzell type **small diffusion problem (I)** .

$$X_n(t) = x_0 + n^{-1/2} \int_{U\times[0,t)} \sigma(X_n(s), u) W(ds \times du) + \int_0^t b(X_n(s))\, ds.$$

- Freidlin-Wentzell type **small diffusion problem (II)** .

$$X_n(t) = x_0 + n^{-1/2} \int_{U\times[0,t)} \sigma_1(X_n(s), u) W(ds \times du) + n^{-1} \int_{U\times[0,t)} \sigma_2(X_n(s), u) \xi(nds \times du) \int_0^t b(X_n(s))\, ds.$$

Examples

- (**LDP for Markov chain:**) Let $\{X_k^n\}$ be a Markov Chain with state space E satisfying

$$X_{k+1}^n = X_k^n + \frac{1}{n} b(X_k^n, \xi_{k+1}),$$

where the ξ_k are iid with distribution π. Want a LDP for $\{X^n(t) \equiv X_{[nt]}^n\}$. Define $\Gamma_n(A,t) = \frac{1}{n}\sum_{k=1}^{[nt]} 1_A(\xi_k)$. and notice that,

$$X^n(t) = X_0^n + \int_{E\times[0,t)} b(X^n(s),u)\Gamma_n(du \times ds)$$

Observe that M_n is a counting measure with mean measure $\pi \otimes \mu_n$, where $\mu_n[0,t] = [nt]$

Examples

- (**Random evolution:**) Suppose $\{\xi_k\}$ is a Markov chain with state space E and

$$X_{k+1}^n = X_k^n + \frac{1}{n} b(X_k^n, \xi_{k+1}).$$

Define $\Gamma_n(A,t) = \frac{1}{n}\sum_{k=1}^{[nt]} 1_A(\xi_k)$.
Then X_n satisfies

$$X_n(t) = X_n(0) + \int_{E\times[0,t)} b(X_n(s), u)\Gamma_n(du \times ds).$$

First step: LDP for stochastic integrals

Suppose $\{Y_n\}$ is a sequence of $\{\mathcal{F}_t^n\}$-adapted semimartingales, $\{X_n\}$ a sequence of adapted cadlag processes, and $\{(X_n, Y_n)\}$ satisfies a LDP.

- **Goal:** To find conditions on the sequence $\{Y_n\}$ such that a LDP holds for $\{(X_{n-} \cdot Y_n)\}$.

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- **Goal:** To find conditions on the sequence $\{Y_n\}$ such that a LDP holds for $\{(X_{n-} \cdot Y_n)\}$.
- In the finite-dimensional case, J. Garcia (2008) described a **uniform exponential tightness** (UET) condition, under which a LDP holds for the tuple $\{(X_n, Y_n, X_{n-} \cdot Y_n)\}$ if $\{(X_n, Y_n)\}$ satisfies a LDP.

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$$J(x, y, z) = \begin{cases} I(x, y), & \text{if } y \text{ is of finite variation, and } z = x \cdot y \\ \infty, & \text{otherwise} \end{cases}$$

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- What happens if Y_n is infinite dimensional?

Infinite dimensional semimartingale: $\mathbb{H}^{\#}$-semimartingale (Kurtz and Protter 1996)

Let $\mathbb{H}$ be a separable Banach space.

Definition

An $\mathbb{R}$-valued stochastic process Y indexed by $\mathbb{H} \times [0, \infty)$ is an $\mathbb{H}^{\#}$-semimartingale with respect to the filtration $\{\mathcal{F}_t\}$ if

- For each $h \in \mathbb{H}$, $Y(h, \cdot)$ is a real valued cadlag $\{\mathcal{F}_t\}$-semimartingale, with $Y(h, 0) = 0$.
- For each $t > 0$ and $h_1, \ldots, h_m \in \mathbb{H}$, and $a_1, \ldots, a_m \in \mathbb{R}$, we have

$$Y(\sum_{i=1}^{m} a_i h_i, t) = \sum_{i=1}^{m} a_i Y(h_i, t) \text{ a.s.}$$

Integration of simple processes w.r.t $\mathbb{H}^{\#}$-semimartingale

Let Z be an $\mathbb{H}$-valued cadlag process of the form

$$Z(t) = \sum_{k=1}^{m} \xi_k(t) h_k, \tag{1}$$

where the $\xi_k(t)$'s are $\{\mathcal{F}_t\}$-adapted real valued cadlag processes and $h_1, \ldots, h_k \in \mathbb{H}$.
Then we define the stochastic integral $Z_- \cdot Y$ in the natural way as

$$Z_- \cdot Y(t) = \sum_{k=1}^{m} \int_0^t \xi_k(s-) dY(h_k, s).$$

Note that the integral above is just a real valued process.

Extension of the stochastic integral for $\mathbb{H}$-valued cadlag processes

Let $\mathcal{S}$ be the collection of all processes of the form (1). Define

$$\mathcal{H}_t = \{\sup_{s \le t} |Z_- \cdot Y(s)| : Z \in \mathcal{S}, \sup_{s \le t} \|Z(s)\| \le 1\}. \tag{2}$$

Definition

An $\mathbb{H}^{\#}$-semimartingale Y is **standard** if for each $t > 0$, $\mathcal{H}_t$ is stochastically bounded, i.e for every $t > 0$ and $\epsilon > 0$ there exists $k(t, \epsilon)$ such that

$$P[\sup_{s \le t} |Z_- \cdot Y(s)| \ge k(t, \epsilon)] \le \epsilon,$$

for all $Z \in \mathcal{S}$, satisfying $\sup_{s \le t} \|Z(s)\| \le 1$.

Examples: Gaussian space-time white noise, Poisson random measure, Banach space-valued semimartingales.

Uniform exponential tightness (Garcia 2008)

Let $\mathcal{S}^n$ denote the space of all piecewise constant, $\{\mathcal{F}_t^n\}$-adapted processes Z, such that $|Z(t)| \leq 1$.

Definition

A sequence of $\{\mathcal{F}_t^n\}$-adapted semimartingales $\{Y_n\}$ is **uniformly exponentially tight** (UET), if for every $t > 0$ and $a > 0$, there exists a $k(t, a)$ such that for all $n > 0$,

$$\limsup_{n\to\infty} n^{-1} \sup_{Z\in\mathcal{S}^n} \log P[\sup_{s\leq t} |Z_- \cdot Y_n(s)| > k(t, a)] \leq -a.$$

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If $E[\sup_{s\leq t} e^{n|Z_- \cdot Y_n(s)|}] \leq e^{nC(t)}$, for all $Z \in \mathcal{S}^n$, $n > 0$, then $\{Y_n\}$ is UET.

UET for sequence of $\mathbb{H}^{\#}$-semimartingales

Let $\mathcal{S}^n$ denote the collection of all $\mathbb{H}$-valued processes Z, such that $\|Z(t)\| \leq 1$ and is of the form

$$Z(t) = \sum_{k=1}^{m} \xi_k(t) h_k,$$

where the ξ_k are cadlag and $\{\mathcal{F}_t^n\}$-adapted $\mathbb{R}$ valued processes and $h_1, \ldots, h_n \in \mathbb{H}$.

Definition

A sequence of $\{\mathcal{F}_t^n\}$ adapted $\mathbb{H}^{\#}$-semimartingales $\{Y_n\}$ is **uniformly exponentially tight**, if for every $a > 0$ and $t > 0$, there exists a $k(t, a)$ such that

$$\limsup_n \frac{1}{n} \sup_{Z \in \mathcal{S}^n} \log P[\sup_{s \leq t} |Z_- \cdot Y_n(s)| > k(t, a)] \leq -a. \qquad (3)$$

Examples of UET sequences

All the integrands mentioned in the examples satisfy UET condition for appropriating indexing Banach spaces.

- If W is a space-time Gaussian white noise on $(E \times [0, \infty), d\mu \times dt)$, then $\{n^{-1/2} W\}$ is UET with the indexing space $\mathbb{H} = L^2(E, \mu)$. That is, define $W(h, t) = \int_{E \times [0,t)} h(x) W(dx, ds)$.

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- Let ξ be a Poisson random measure on $U \times [0,\infty)$ with mean measure $d\nu \times dt$ and define $\xi_n(A,t) = n^{-1}\xi(A,nt)$. $\{\xi_n\}$ forms a UET sequence with the indexing space $\mathbb{H} = L^\Phi(\nu)$ with $\Phi(x) = e^x - 1$. Here $L^\Phi(\nu)$ is the Orlicz space with respect to Φ.

Definition of LDP for a sequence of $\mathbb{H}^{\#}$-semimartingale

Definition

Let $\{Y_n\}$ be a sequence of $\{\mathcal{F}_t^n\}$-adapted $\mathbb{H}^{\#}$-semimartingales and $\{X_n\}$ be a sequence of cadlag, $\{\mathcal{F}_t^n\}$-adapted $\mathbb{H}$-valued processes. Let A denote the index set consisting of all ordered finite subsets of $\mathbb{H}$. $\{(X_n, Y_n)\}$ is said to satisfy a **large deviation principle** with the rate function family $\{I_\alpha : \alpha \in A\}$ if for $\alpha = (\phi_1, \ldots, \phi_k)$, $\{(X_n, Y_n(\phi_1, \cdot), \ldots, Y_n(\phi_k, \cdot))\}$ satisfies LDP in $D_{\mathbb{H}\times\mathbb{R}^k}[0, \infty)$ with rate function I_α.

LDP theorem

Theorem

Let $\{Y_n\}$ *be a sequence of* $\mathbb{H}^{\#}$*-semimartingales and* $\{X_n\}$ *be a sequence of cadlag, adapted* $\mathbb{H}$*-valued processes. Assume* $\{Y_n\}$ *is UET. If* $\{(X_n, Y_n)\}$ *satisfies LDP in the sense of Definition 6, with the rate function family* $\{I_\alpha : \alpha \in A\}$ *, then* $\{(X_n, Y_n, X_{n-} \cdot Y_n)\}$ *also satisfies a LDP.*

Identification of the rate function

For any separable Banach space $\mathbb{H}$, there exists a sequence $\{(\phi_k, p_k) : \phi_k \in \mathbb{H}, p_k \in C(\mathbb{H}, \mathbb{R})\}$ satisfying some nice properties such that

$$h = \sum_k p_k(h)\phi_k.$$

Let $I(x, y_1, y_2, \ldots)$ be the rate function of $\{(X_n, Y_n(\phi_1, \cdot), Y_n(\phi_2, \cdot), \ldots)\}$ in $D_{\mathbb{H}\times\mathbb{R}^\infty}[0, \infty)$.
Put $\mathbf{y}^*(t) = \sum_k y_k(t)p_k$.

If $I(x, y_1, y_2, \ldots) < \infty$, then $\mathbf{y}^*$ exists, $\mathbf{y}^* \in D_{\mathbb{H}^*}[0, \infty)$ and $T_t(\mathbf{y}^*) < \infty$, where the total variation $T_t(\mathbf{y}^*)$ is defined as

$$T_t(\mathbf{y}^*) = \sup_\sigma \sum_i \|\mathbf{y}^*(t_i) - \mathbf{y}^*(t_{i-1})\|_{\mathbb{H}^*},$$

the sup being taken over all partitions $\sigma \equiv \{t_i\}_i$ of $[0, t)$.
Consequently, for $x \in D_{\mathbb{H}}[0, \infty)$,
$x \cdot \mathbf{y}^*(t) = \lim \sum_i \langle x(t_i), y(t_{i+1}) - y(t_i) \rangle_{\mathbb{H},\mathbb{H}^*}$ exists.
Let $\mathcal{D} = \{\mathbf{y}^* \in D_{\mathbb{H}^*}[0, \infty) : T_t(\mathbf{y}^*) < \infty \text{ for all } t > 0\}$

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Theorem

The rate function for $\{(X_n, Y_n(\phi_1, \cdot), Y_n(\phi_2, \cdot), \ldots, X_{n-} \cdot Y_n)\}$ *is given by*

$$J(x, y_1, y_2, \ldots, z) = \begin{cases} I(x, y_1, y_2, \ldots), & \text{if } \mathbf{y}^* \in \mathcal{D} \text{ and } z = x \cdot \mathbf{y}^* \\ \infty, & \text{otherwise} \end{cases}$$

Stochastic differential equation

Let $F, F_n : \mathbb{R}^d \to \mathbb{H}$ be measurable functions and X_n satisfy

$$X_n(t) = U_n(t) + F_n(X_{n-}) \cdot Y_n(t).$$

Assume that

- $\{Y_n\}$ is UET and $\{(U_n, Y_n)\}$ satisfies LDP
- For all x whenever $x_n \to x$, $F_n(x_n) \to F(x)$.
- $\{(U_n, X_n, Y_n)\}$ is exponentially tight
- Suppose that for every $(u, y_1, y_2, \ldots) \in D_{\mathbb{R}^d \times \mathbb{R}^\infty}[0, \infty)$ for which $I_\alpha(u, y_1, y_2, \ldots) < \infty$, the solution to

$$x = u + F(x) \cdot \mathbf{y}^*$$

is unique.

Then the sequence $\{(U_n, X_n, Y_n(\phi_1, \cdot), Y_n(\phi_2, \cdot), \ldots)\}$ satisfies a LDP with the rate function given by

$$J(u, x, y_1, y_2, \ldots) = \begin{cases} I(u, y_1, y_2, \ldots), & x = u + F(x) \cdot \mathbf{y}^*, \quad \mathbf{y}^* \in \mathcal{D}. \\ \infty, & \text{otherwise.} \end{cases}$$