Tightness of Maxima of Generalized Branching Random Walks

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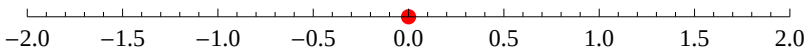
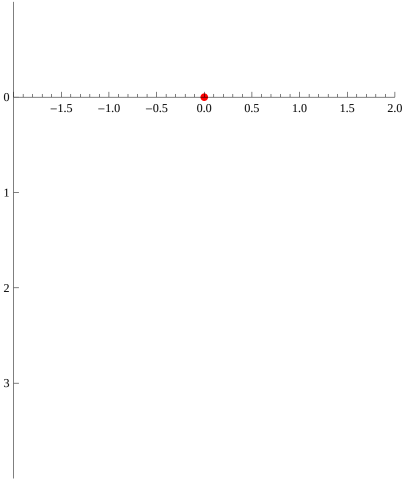
Outline of the talk

- The Theorem.
  - Generalized Branching Random Walk.
  - Tightness.
- The Proof.
  - Recursion and Recursion Inequalities.
  - Exponential Tail and Lyapunov Function.
  - 'Flatness' Argument.

Branching Random Walk





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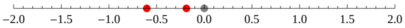

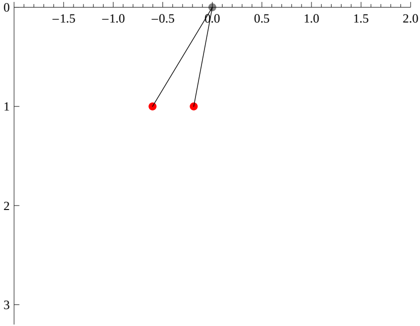

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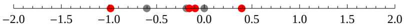

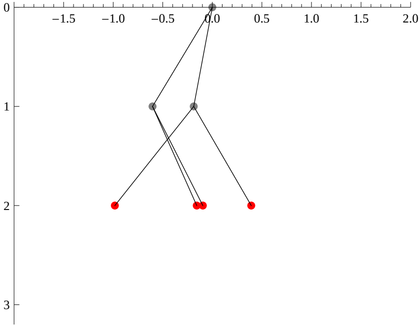

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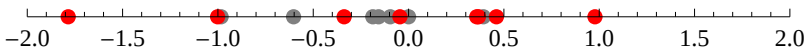

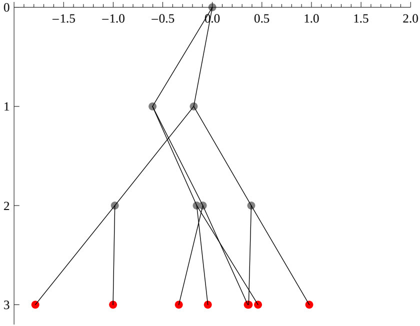

Branching Random Walks

Randomness is governed by two families of distributions:

- $\{p_{n,k}\}_{k=1}^{\infty}$: branching laws on $\mathbb{N}$ at time n.
- $G_{n,k}(x_1, \ldots, x_k)$: joint distribution functions on $\mathbb{R}^k$ at time n.

Tightness of (Shifted) Maxima Distributions

Let M_n be the maximal displacement of particles at level n and $F_n(\cdot)$ the distribution of M_n, then

Theorem

The family of shifted distributions $F_n(\cdot - Med(F_n))$ is tight under mild assumptions.

On the branching laws,

(B1) $\{p_{n,k}\}_{n\geq 0}$ possess a uniformly bounded support, i.e., there exists an integer $k_0 > 1$ such that $p_{n,k} = 0$ for all n and $k \notin \{1, \ldots, k_0\}$.

(B2) The mean offspring number is uniformly greater than 1 by some fixed constant. I.e., there exists a real number $m_0 > 1$ such that $\inf_n\{\sum_{k=1}^{k_0} kp_{n,k}\} > m_0$.

Assumptions on the walk.

On the step distributions,

(G1) There exist $a > 0$ and $M_0 > 0$ such that $\bar{g}_{n,k}(x+M) \leq e^{-aM}\bar{g}_{n,k}(x)$ for all n, k and $M > M_0$, $x \geq 0$.

(G2) For some fixed $\epsilon_0 < \frac{1}{4}\log m_0 \wedge 1$, there exists an x_0 such that $\bar{g}_{n,k}(x_0) \geq 1 - \epsilon_0$ for all n and k, where $\bar{g}_{n,k}(x) = 1 - g_{n,k}(x)$. By shifting, we may and will assume that $x_0 = 0$, that is, $\bar{g}_{n,k}(0) \geq 1 - \epsilon_0$.

(G3) For any $\eta_1 > 0$, there exists a $B > 0$ such that $G_{n,k}(B, \ldots, B) \geq 1 - \eta_1$ and $G_{n,k}([-B, \infty)^k) \geq 1 - \eta_1$ for all n and k.

Recursion on $\bar{F}_{n+1}$ and $\bar{F}_n$.

Assume: binary branching and no time dependence. (Dependence between siblings still exists.)
We have the following recursion,

$$F_{n+1}(x) = \int_{\mathbb{R}^2} F_n(x - y_1) F_n(x - y_2) d^2 G(y_1, y_2).$$

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Consider $\bar{F}_n(x) = 1 - F_n(x)$, the recursion becomes

$$\bar{F}_{n+1}(x) = 1 - \int_{\mathbb{R}^2} (1 - \bar{F}_n(x - y_1))(1 - \bar{F}_n(x - y_2)) d^2 G(y_1, y_2).$$

Bounds (Inequalities) on Recursion

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Using $(1 - x_1)(1 - x_2) \geq 1 - x_1 - x_2$, one has

$$\bar{F}_{n+1}(x) \leq \int_{\mathbb{R}} 2\bar{F}_n(x - y) dg(y).$$

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Using Hölder's inequality, one has

$$\bar{F}_{n+1}(x) \geq \int_{\mathbb{R}} \left(2\bar{F}_n(x - y) - (\bar{F}_n(x - y))^2\right) dg(y).$$

Lyapunov Function and Exponential Tail

In order to prove the tightness of $F_n(\cdot - Med(F_n))$, it is sufficient to prove an exponential decay uniform in n. I.e., for fixed $\eta_0 \in (0,1)$, there exist an $\hat{\epsilon}_0 = \hat{\epsilon}_0(\eta_0) > 0$, an n_0 and an $\hat{M}$ such that, if $n > n_0$ and $\bar{F}_n(x - \hat{M}) \leq \eta_0$, then

$$\bar{F}_n(x - \hat{M}) \geq (1 + \hat{\epsilon}_0)\bar{F}_n(x).$$

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For this reason, we introduce a Lyapunov function

$$L(u) = \sup_{\{x: u(x) \in (0, \frac{1}{2}]\}} l(u; x),$$

where

$$l(u; x) = \log\left(\frac{1}{u(x)}\right) + \log_b\left(1 + \epsilon_1 - \frac{u(x-M)}{u(x)}\right)_+ .$$

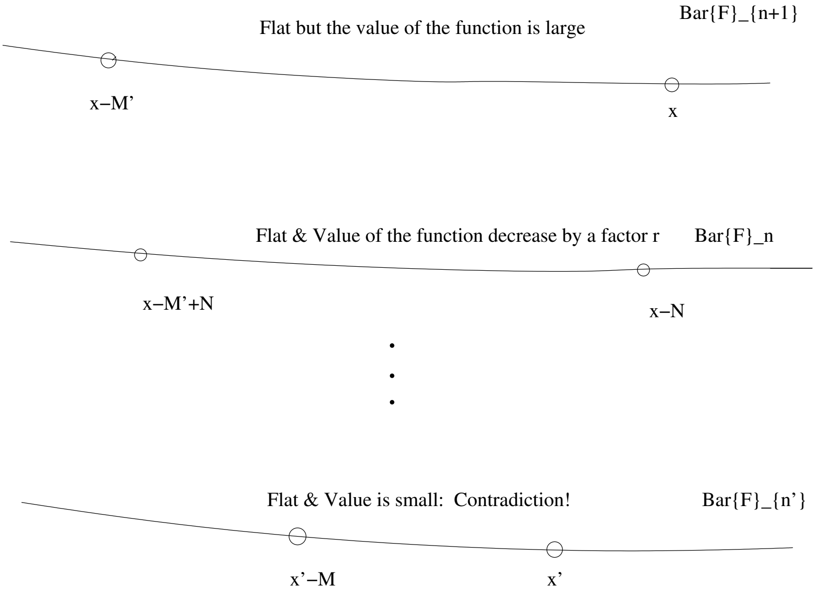
Here $(x)_+ = x \vee 0$ and $\log 0 = -\infty$.

Why Choose This Lyapunov Function

Suppose that $\sup_n L(\bar{F}_n) < C$ for some constant $C > 0$ (IOU!), then there exists a δ_1 such that, for all n,

$$\bar{F}_n(x) \leq \delta_1 \text{ implies } \bar{F}_n(x - M) \geq (1 + \frac{\epsilon_1}{2})\bar{F}_n(x).$$

This is not quite the exponential decay we need. But the tightness of G implies the following (pictures in the next slide) which closes the gap.


Flat but the value of the function is large
Bar{F}_{n+1}
x−M'
x
Flat & Value of the function decrease by a factor r
Bar{F}_n
x−M'+N
x−N
Flat & Value is small: Contradiction!
Bar{F}_{n'}
x'−M
x'

Simplify the Notation

$\sup_n L(\bar{F}_n) < C$ (IOU) follows from

Suppose that two non-increasing cadlag function $u, v : [0,1] \to [0,1]$ satisfy

$$\int_{\mathbb{R}} (2u - u^2)(x-y)dg(y) \leq v(x) \leq \int_{\mathbb{R}} 2u(x-y)dg(y).$$

Then $L(v) > C$ implies $L(u) > L(v)$.

To proof this claim, we need to compare the value of u and v and the flatness of u and v.

Information on v

Since $L(v) > C$, by definition, one can find (let $x_2 := x_1 - M$) that

$$1 + \epsilon := \frac{v(x_2)}{v(x_1)} < 1 + \epsilon_1$$

and

$$v(x_1) < (\epsilon_1 - \epsilon)^{1/\log b} e^{-C} < \frac{1}{2}. \quad \textbf{Tiny!}$$

Want: $u(x_1')$ small and flat (i.e., $\frac{u(x_1' - M)}{u(x_1')}$ is small).

The analysis of u

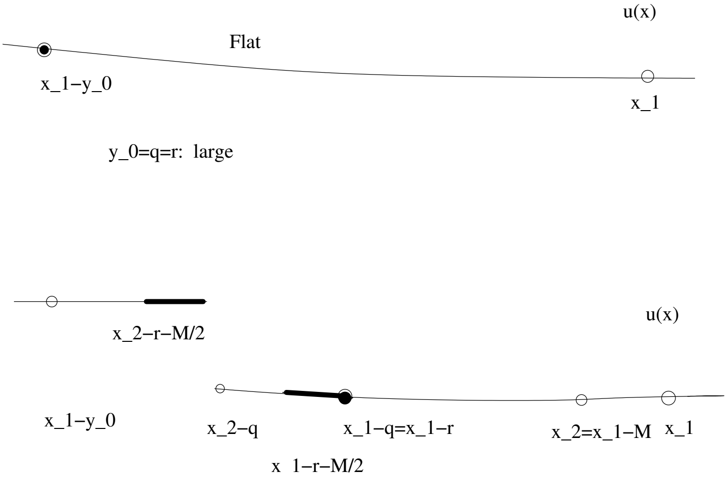
An application of Chebyshev inequality shows that

$$u(x_1) \leq u(x_2) \leq \frac{1+\epsilon}{m_0(1-\epsilon_0)} v(x_1)$$

is tiny. But we cannot get any information about flatness of u at x_1. In order to search for a flat piece of u, we start from $v(x_2) = (1+\epsilon)v(x_1)$, which implies that

$$\int_{\mathbb{R}} (2u - u^2)(x_2 - y) dg(y) \leq (1+\epsilon) \int_{\mathbb{R}} 2u(x_1 - y) dg(y).$$

Flatness of u —- truncation argument



Step1: With the previous notations, we can truncate the integral at an affordable cost $\delta = \kappa(\epsilon_1 - \epsilon)$, i.e.,

$$\int_{-\infty}^{r} (2u - u^2)(x_2 - y)dg(y) \leq (1 + \epsilon + \delta) \int_{-\infty}^{r} 2u(x_1 - y)dg(y).$$

Step2: Since $u(x_1)$ is tiny, $u(x_1 - r)$ is very small even if the value of u increases slowly for a long time compared with $u(x_1)$. Therefore, the nonlinear term is negligible with another affordable cost, i.e.,

$$\int_{-\infty}^{r} u(x_2 - y)dg(y) \leq (1 + \epsilon + 2\delta) \int_{-\infty}^{r} u(x_1 - y)dg(y).$$

Step3: From step 2, we can find a location in $[x_1 - r, \infty)$ where u is flat. In fact we need the following a stronger version.

(a) $u(x_2 - y_1) \leq (1 + \epsilon + 3\delta)u(x_1 - y_1)$ for some $y_1 \leq r' \wedge M$, or

(b) $u(x_2 - y_1) \leq (1 + \epsilon + 2\delta - \delta e^{ay_1/8})u(x_1 - y_1)$ for some $y_1 \in (M, r]$.

The End

Thank You!