Homework 3: Exercises on Brownian motion and diffusions
Due April 22, 2020

**Problem 4.1.a.** Write a code that generates Brownian motion on the interval \([0, T]\), where \(T\) is a parameter to be set at the beginning of the code. (So write the code with a general \(T\) and at the beginning of the code set \(T = 5\).) For this use the approximation with random walk. Precisely, let \(N\) be the resolution parameter (an integer you set at the beginning of the code) and then \(B(0) = 0\) and for an integer \(k \in \{1, \ldots, \lfloor NT \rfloor\}\), \(B(k/N)\) is given approximately by \((S_k - k\mu)/(\sigma \sqrt{N})\), where \(S_k = X_1 + \cdots + X_k\) and \(X_k\) are i.i.d. random variables with mean \(\mu\) and variance \(\sigma^2\). Plot \(B(t)\) as a function of \(t\) (by plotting \(B(k/N)\) against \(k/N\) and joining the dots). Vary the distribution of the increments of the walk and see how that affects your approximation. Vary the size of \(N\) (e.g. \(N \in \{10, 100, 1000, 10000\}\)) and see how that affects the plot.

**Problem 4.1.b.** Use the above code to generate two independent Brownian motions on the interval \([0, T]\) then plot the two coordinates against each other to see the path of a two-dimensional Brownian motion. Repeat for three-dimensional Brownian motion. Plot several samples of each and see if you can observe the recurrence and transience properties of the curve. (For this you need to use a large \(T\).)

**Problem 4.1.c.** Use the above code to estimate the expected values of \(\max_{0 \leq t \leq 1} B(t)\), \(\int_0^2 B(t) \, dt\), and the Lebesgue measure of the set \(\{t \in [0, 4] : B(t) > 0\}\). Using the reflection principle, compute the expected value of \(\max_{0 \leq t \leq 1} B(t)\) and compare with your numerical result.

**Problem 4.1.d.** Use the above code to solve the equation \(\Delta u = 0\) on the square \((0, 1)^2\) the the boundary conditions \(u(0, y) = 0, u(1, y) = 1\) for \(0 < y < 1\), and \(u(x, 0) = 1, u(x, 1) = 2\) for \(0 < x < 1\). (The values at the corners are immaterial. Why?) Compute \(u(i/N, j/N)\) for say \(N = 1000\) (better set it as a parameter at the beginning of your code) and \(i, j\) varying from 0 to \(N\). Plot the solution \(u\) on the square \([0, 1]^2\) using a color-coded plot (red should mean hotter, i.e. higher values of \(u\), and blue should mean colder, i.e. lower values of \(u\)). Does the plot look like what you expected? Play around with other boundary conditions!
Problem 4.2. Prove that $B(t)$ and $B(t)^2 - t$ are martingales (in the natural filtration of the Brownian motion).

Problem 4.3. Consider a real-valued Markov process with infinitesimal generator $L$. Recall that $\mu$ is an invariant measure if

$$\int E_x[f(B(t))] \mu(dx) = \int f(x) \mu(dx)$$

for any bounded measurable function $f$ and all $t > 0$. Prove that

$$\int Lf(x) \mu(dx) = 0,$$

for functions $f$ in the domain of $L$.

Problem 4.4. Recall that if $T$ is the first time Brownian motion exits the interval $[0, 1]$, then $u(t, x) = P_x(T > t)$ solves the heat equation $\partial_t u = \partial_{xx} u/2$ on $(0, 1)$ with initial condition $u(0, x) = 1$ for $x \in (0, 1)$ and boundary conditions $u(t, 0) = u(t, 1) = 0$ for all $t > 0$. We would like to get a formula for $u(t, x)$.

First, for each fixed $t \geq 0$, extend $u$ to an odd function on $[-1, 1]$ by defining $u(t, -x) = -u(t, x)$ for $x \in [-1, 0]$. The theory of Fourier series tells us then that if we define for $k \in \mathbb{N}$

$$a_k(t) = \int_{-1}^{1} u(t, x) \sin(\pi k x) \, dx = 2 \int_{0}^{1} u(t, x) \sin(\pi k x) \, dx,$$

then

$$u(t, x) = \sum_{k \geq 1} a_k(t) \sin(\pi k x).$$

Prove that $a'_k(t) = -\pi^2 k^2 a_k(t)$ for all $t > 0$ and $k \in \mathbb{N}$. Compute $a_k(0)$? Now find $a_k(t)$ for $t > 0$ and $k \in \mathbb{N}$ which then gives you a formula for $u(t, x)$.

Now shift, rescale, and choose $x$ appropriately to get a formula for $P_0(T_{a,b} > t)$, where $T_{a,b}$ is the first exit time from the interval $[a, b]$ with $a < 0 < b$. Use this to compute $P_0(\sup_{0 \leq t \leq 1} |B(t)| \geq x)$. (Note: $t$ and $x$ here do not have much to do with $t$ and $x$ above!)

Problem 4.5. Compute $\int_{0}^{t} B(s) \, dB(s)$ analytically then simulate your solution. Using the same Brownian motion you used in the simulation (not
a new sample), compute the integral numerically using the left-, mid-, and right-point rules. Plot all four solutions. What do you notice?

**Problem 4.6.a.** Solve the stochastic differential equation (SDE)

\[ dX_t = \mu X_t \, dt + \sigma X_t \, dB_t \]

with initial condition \( X_0 = x \), where \( \mu \in \mathbb{R} \) and \( \sigma > 0 \) are some given constants. To do so, first try to use separation of variables, i.e. divide by \( X_t \) and integrate out, then solve for \( X_t \). Apply Itô’s formula to the \( X_t \) that you got this way. Do you get back the above SDE? Stare at what you got and try to guess how to fix the formula for \( X_t \) so that it DOES solve the SDE. This is called geometric Brownian motion and is used in the Black-Scholes model of stock prices.

**Problem 4.6.b.** Solve the SDE from the above problem numerically, say with initial condition \( x = 0 \) and parameters \( \mu = \sigma = 1 \). Then using the same Brownian motion you used for the numerical solution (not a new sample) simulate the analytic solution you found. How close is your numerical solution to the simulation of the analytical solution?

**Problem 4.7.a.** Solve the SDE

\[ dX_t = a(\mu - X_t) \, dt + \sigma dB_t \]

with initial condition \( X_0 = x \). Here, \( \mu \in \mathbb{R} \), \( a > 0 \), and \( \sigma > 0 \) are given parameters. To solve the equation consider \( f(t, X_t) = X_t e^{at} \) and use the method of variation of parameters to show that

\[ X_t = xe^{-at} + \mu(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} \, dB_s. \]

This is the Ornstein-Uhlenbeck process. It is an example of a noisy relaxation process (check out what it does when \( \sigma = 0 \)). It is used in financial and physical sciences. E.g. when \( \mu = 0 \) it models a Brownian particle with diffusion \( \sigma \), but undergoing friction.

**Problem 4.7.b.** Solve the SDE from the above problem numerically, say with initial condition \( x = 1 \) and parameters \( \mu = 0 \) and \( a = \sigma = 1 \). Then using the same Brownian motion you used for the numerical solution (not a new sample) simulate the analytic solution (??). How close is your numerical solution to the simulation of the analytical solution?