NOTE: You only need to solve the underlined problems. The others are recommended, but not required. The starred problems are ones that are recommended for students who have already taken graduate probability.

**Problem 3.1.** Consider the following queuing model: Fix \( p, q \in (0, 1) \). Every minute a customer arrives at the post office with probability \( p \) and goes to the end of the line at the only open register. Also, every minute, with probability \( q \) the employee at the register serves the customer at the top of the line, if there is one. Let \( X_n \) be the number of customers in line, at minute \( n \). Initially, \( X_0 = 0 \) (the post office just opened). This is a Markov chain. What is its state space? What are the transition probabilities? When is the chain transient, null recurrent, positive recurrent? **Hint:** This is a birth-death process, but with a some probability for staying at the same site. So you need to rework the conditions we worked out in class.

**Problem 3.2.** In class, I gave two solutions of
\[
f(T^{-1} \omega)p_{-1}(\omega) + f(T \omega)(1 - p_1(\omega)) = f(\omega)
\]
as
\[
f(\omega) = \frac{1}{p_0} \sum_{i=0}^{\infty} \prod_{j=1}^{i} \rho_j \quad \text{and} \quad f(\omega) = \frac{1}{1 - p_0} \sum_{i=0}^{\infty} \prod_{j=1}^{i} \rho_{-j}^{-1},
\]
where \( \rho_j = (1 - p_j)/p_j \). Verify these do satisfy the equation.

**Problem 3.3.** In class, we found that one-dimensional nearest-neighbor random walk in i.i.d. random environment satisfies \( P(S_n \to \infty) = 1 \) if \( \mathbb{E}[\log \rho] < 0 \) and that then the walk satisfies the law of large numbers \( P(S_n/n \to v) = 1 \) with \( v = (1 - \mathbb{E}[\rho])/(1 + \mathbb{E}[\rho]) \). Use the simple model where \( p_0 \) only takes two values, \( q \) with probability \( \varepsilon \in (0, 1/2) \), and \( p \) with probability \( 1 - \varepsilon \), to show that one can have \( \mathbb{E}[\log \rho] < 0 \) and yet \( v = 0 \).

**Problem 3.4.a.** Fix \( \lambda > 0 \). Fix a positive integer \( n \). A coin that comes up heads with probability \( p = \lambda/n \) is tossed repeatedly. Let \( X_n \) be the random variable indicating the number of tosses before the first head occurred. Prove that as \( n \to \infty \), \( X_n \) converges weakly to an exponential
random variable with parameter $\lambda$, i.e. to a random variable with CDF $P(X \leq x) = 1 - e^{-\lambda x}$.

**Note:** This should convince you that exponential random variables are reasonable models for waiting times.

**Problem 3.4.b** Let $X$ be an exponential random variable with rate $\lambda$. Prove that $E[X] = 1/\lambda$.

**Note:** By the law of large numbers, this implies that if $X_n$ are i.i.d. exponential random variables with rate $\lambda$, then $(X_1 + \cdots + X_n)/n \to 1/\lambda$ almost surely. If you think of $X_1 + \cdots + X_n$ as the time it took to serve $n$ customers, then $n/(X_1 + \cdots + X_n)$ is the rate at which the service is being performed. Hence, the parameter $\lambda$ is called the rate of the exponential random variable.

**Problem 3.4.c.** Let $X$ and $Y$ be two independent exponential random variables with rates $\lambda$ and $\gamma$, respectively. Prove that $\min(X,Y)$ is an exponentially distributed random variable with rate $\lambda + \gamma$.

**Note:** This shows again that modeling waiting times with exponential random variables goes along with our intuition: say you are in a bank and there are two tellers. You are waiting for the first one to be available. Your waiting time is the minimum of the two times at which each of the tellers will become free. Furthermore, the number of customers these two tellers will serve in a unit time is the sum of the two rates.

**Problem 3.4.d** For $\alpha, \lambda > 0$, a gamma($\alpha, \lambda$) distributed random variable is one that has pdf $f(x) = C_{\alpha,\lambda} x^{\alpha-1} e^{-\lambda x}$ when $x > 0$ and $f(x) = 0$ otherwise. Find the constant $C_{\alpha,\lambda}$ that makes this a pdf. Prove that the sum of two independent gamma($\alpha, \lambda$) and gamma($\beta, \lambda$) is a gamma($\alpha + \beta, \lambda$). **Hint:** The pdf of two independent random variables with pdfs $f$ and $g$ is given by $h(x) = \int f(y)g(x - y)dy$. Though be careful with when the pdfs are 0.

**Note:** This shows two things. First, that the sum of $n$ independent exponential random variables with rate $\lambda$ each is a gamma($n, \lambda$). And second, gamma random variables could also be used as models for waiting times, if e.g. we believe that the waiting process involves a few consecutive waits at the same rate. E.g. commute times, where there is a known number of traffic lights on the way.
Problem 3.5. Fix a $\lambda > 0$. There are $n$ people in a supermarket. Each decides with probability $p = \lambda/n$ to go check out. A line forms. Let $X_n$ be the number of people in the line. This is a binomial random variable with parameters $n$ and $p$ and has an average of $\lambda$. Prove that $X_n$ converges weakly to a Poisson random variable with parameter $\lambda$, i.e. a $\mathbb{Z}_+$-valued random variable with mass function given by $f(k) = e^{-\lambda}\lambda^k/k!$, $k \in \mathbb{Z}_+$. Show that the mean of this random variable is $\lambda$.

Note: This shows you that using Poisson random variables is a good model for the length of a queue.

Problem 3.6.a. Suppose $N(t)$ is a simple jump process (aka an arrival process, i.e. a piecewise-constant càdlàg function that only has jumps of size 1) that has independent and stationary increments and satisfies $N(0) = 0$. Let $p(t) = P(N(t) = 0)$, i.e. the probability that there were no jumps by time $t$. Prove that $p(t + s) = p(t)p(s)$. Conclude the existence of a $\lambda \geq 0$ such that $p(t) = e^{-\lambda t}$.

Problem 3.6.b. The case $\lambda = 0$ is not interesting (why not?). So we will assume $\lambda > 0$. Let $T_k$, $k \geq 1$, denote the jump times of $N(t)$. Set $T_0 = 0$. Let $\tau_k = T_k - T_{k-1}$ for $k \in \mathbb{N}$. The previous problem proved that $P(\tau_1 > t) = e^{-\lambda t}$ (how is that?), i.e. that $\tau_1$ is exponentially distributed. Prove that $E[1\{\tau_{k+1} > t\}|T_k] = e^{-\lambda t}$ almost surely. (You will need to prove that since $T_k$ is a stopping time for the filtration $\mathcal{F}_t = \sigma(N(s) : 0 \leq s \leq r)$, the stationary and independent increments property implies that $N(T_k + t) - N(T_k)$ is independent of the stopped filtration $\mathcal{F}_{T_k}$ and has the same distribution as $N(t)$.) Deduce that $\{\tau_k : k \in \mathbb{Z}_+\}$ are i.i.d. exponential random variables with rate $\lambda$.

Note: This shows that a Poisson process is the unique simple jump process with stationary and independent increments! (In contrast, we will see later that a Brownian motion is the unique continuous process with stationary and independent increments!!)

For the next question we need to define order statistics. Given a set of numbers $X_1, \ldots, X_n$ we let $X_{1:n} = \min(X_1, \ldots, X_n)$, $X_{2:n}$ the second-smallest number (which could equal $X_{1:n}$ if there are two smallest numbers), etc. So $X_{n:n} = \max(X_1, \ldots, X_n)$.

Problem 3.7. Let $\tau_1, \ldots, \tau_{n+1}$ be i.i.d. exponential random variables with rate $\lambda$. Let $t > 0$. Prove that conditional on $\{\tau_1 + \cdots + \tau_n \leq t < \tau_1 + \cdots + \tau_{n+1}\}$
The distribution of \((\tau_1, \tau_1 + \tau_2, \ldots, \tau_1 + \cdots + \tau_n)\) is the same as that of \((U_{1:n}, U_{2:n}, \ldots, U_{n:n})\) where \((U_1, \ldots, U_n)\) are i.i.d. uniform on \([0, t]\).

More problems soon....