Homework 2: Exercises on Markov Processes
Due April 8, 2020

Problem 3.1. Consider the following queuing model: Fix $p, q \in (0, 1)$. Every minute a customer arrives at the post office with probability $p$ and goes to the end of the line at the only open register. Also, every minute, with probability $q$ the employee at the register serves the customer at the top of the line, if there is one. Let $X_n$ be the number of customers in line, at minute $n$. Initially, $X_0 = 0$ (the post office just opened). This is a Markov chain. What is its state space? What are the transition probabilities? When is the chain transient, null recurrent, positive recurrent? **Hint:** This is a birth-death process, but with a some probability for staying at the same site. So you need to rework the conditions we worked out in class.

Problem 3.2. In class, I gave two solutions of

$$f(T^{-1}\omega)p_{-1}(\omega) + f(T\omega)(1 - p_1(\omega)) = f(\omega)$$

as

$$f(\omega) = \frac{1}{p_0} \sum_{i=0}^{\infty} \prod_{j=1}^{i} \rho_j \quad \text{and} \quad f(\omega) = \frac{1}{1 - p_0} \sum_{i=0}^{\infty} \prod_{j=1}^{i} \rho_j^{-1},$$

where $\rho_j = (1 - p_j)/p_j$. Verify these do satisfy the equation.

Note: The rest of the exercises are elementary. Their point was to set the ground for questions about Poisson processes and continuous-time discrete-space Markov chains. Those got split into a separate homework set which is not currently due, but will be posted as HW2b for now, in case you still would like to work on them.

Problem 3.3. In class, we found that one-dimensional nearest-neighbor random walk in i.i.d. random environment satisfies $P(S_n \to \infty) = 1$ if $\mathbb{E}[\log \rho] < 0$ and that then the walk satisfies the law of large numbers $P(S_n/n \to v) = 1$ with $v = (1 - \mathbb{E}[\rho])/(1 + \mathbb{E}[\rho])$. Use the simple model where $p_0$ only takes two values, $q$ with probability $\varepsilon \in (0, 1/2)$, and $p$ with probability $1 - \varepsilon$, to show that one can have $\mathbb{E}[\log \rho] < 0$ and yet $v = 0$.

Problem 3.4.a. Fix $\lambda > 0$. Fix a positive integer $n$. A coin that comes up heads with probability $p = \lambda/n$ is tossed repeatedly. Let $X_n$ be the
random variable indicating the number of tosses before the first head occurred. Prove that as \( n \to \infty \), \( X_n \) converges weakly to an exponential random variable with parameter \( \lambda \), i.e. to a random variable with CDF \( P(X \leq x) = 1 - e^{-\lambda x} \).

**Note:** This should convince you that exponential random variables are reasonable models for waiting times.

**Problem 3.4.b** Let \( X \) be an exponential random variable with rate \( \lambda \). Prove that \( E[X] = 1/\lambda \).

**Note:** By the law of large numbers, this implies that if \( X_n \) are i.i.d. exponential random variables with rate \( \lambda \), then \( (X_1 + \cdots + X_n)/n \to 1/\lambda \) almost surely. If you think of \( X_1 + \cdots + X_n \) as the time it took to serve \( n \) customers, then \( n/(X_1 + \cdots + X_n) \) is the rate at which the service is being performed. Hence, the parameter \( \lambda \) is called the rate of the exponential random variable.

**Problem 3.4.c.** Let \( X \) and \( Y \) be two independent exponential random variables with rates \( \lambda \) and \( \gamma \), respectively. Prove that \( \min(X,Y) \) is an exponentially distributed random variable with rate \( \lambda + \gamma \).

**Note:** This shows again that modeling waiting times with exponential random variables goes along with our intuition: say you are in a bank and there are two tellers. You are waiting for the first one to be available. Your waiting time is the minimum of the two times at which each of the tellers will become free. Furthermore, the number of customers these two tellers will serve in a unit time is the sum of the two rates.

**Problem 3.4.d** For \( \alpha, \lambda > 0 \), a gamma(\( \alpha, \lambda \)) distributed random variable is one that has pdf \( f(x) = C_{\alpha, \lambda} x^{\alpha-1} e^{-\lambda x} \) when \( x > 0 \) and \( f(x) = 0 \) otherwise. Find the constant \( C_{\alpha, \lambda} \) that makes this a pdf. Prove that the sum of two independent gamma(\( \alpha, \lambda \)) and gamma(\( \beta, \lambda \)) is a gamma(\( \alpha + \beta, \lambda \)). **Hint:** The pdf of two independent random variables with pdfs \( f \) and \( g \) is given by \( h(x) = \int f(y)g(x-y)dy \). Though be careful with when the pdfs are 0.

**Note:** This shows two things. First, that the sum of \( n \) independent exponential random variables with rate \( \lambda \) each is a gamma(\( n, \lambda \)). And second, gamma random variables could also be used as models for waiting times, if e.g. we believe that the waiting process involves a few consecutive waits at
the same rate. E.g. commute times, where there is a known number of traffic 
lights on the way.

**Problem 3.5.** Fix a $\lambda > 0$. There are $n$ people in a supermarket. Each 
decides with probability $p = \lambda/n$ to go check out. A line forms. Let $X_n$ 
be the number of people in the line. This is a binomial random variable 
with parameters $n$ and $p$ and has an average of $\lambda$. Prove that $X_n$ converges 
weakly to a Poisson random variable with parameter $\lambda$, i.e. a $\mathbb{Z}_+$-valued 
random variable with mass function given by $f(k) = e^{-\lambda} \lambda^k / k!$, $k \in \mathbb{Z}_+$. 
Show that the mean of this random variable is $\lambda$.

**Note:** This shows you that using Poisson random variables is a good 
model for the length of a queue.