Homework 1: Due Tuesday January 29, 2019

1. Solve exercises 1.1, 1.2, 1.3 from the textbook.

2. Let X_n be the discrete uniform on $\{1/2^n, 2/2^n, 3/2^n, \cdots, (2^n-1)/2^n, 2^n/2^n\}$. Prove that X_n converges in distribution to a Uniform(0,1).

This shows how to approximate a uniform distribution using coin tosses (how?!).

3. Fix $\lambda > 0$. For $n > \lambda$ let X_n be Binomial$(n, \lambda/n)$. Prove that X_n converges in distribution to a Poisson(λ). (Hint: compute the moment generating function of a Bernoulli(p). Deduce the moment generating function of X_n. Take $n \to \infty$. Compute the moment generating function of a Poisson(λ) and compare with the limit.)

This result is called the law of rare events. It shows why a Poisson(λ) is a good model for the length of a queue: if you have a finite population of n people and you ask each of them, independently, to join the queue with a small probability of λ/n, then X_n is the length of the queue. Thus, when the size of the population grows large (and the probability each individual joins the queue becomes small, but in such a way that the average number of people joining the queue remains λ), the length of the queue becomes a Poisson(λ) random variable.

4. Fix $\lambda > 0$. For $n > \lambda$ let X_n be Geometric(λ/n). Show that X_n/n converges in distribution to an Exponential(λ). (Hint: again, compute moment generating functions.)

This result shows why exponential random variables are good models for waiting times: Say you are waiting to be served and the cashier decides when to serve you by tossing a coin every $1/n$ time unit. The coin comes up heads with probability λ/n. The cashier will serve you as soon as they get their first head. Then X_n is the number of coin tosses it will take you to be served and X_n/n is the corresponding amount of time. Taking $n \to \infty$ means the time resolution is being taken to 0, to turn the model into a continuous-time one instead of a discrete-time model. As a result, the waiting time becomes an Exponential(λ). For the geometric random variable, λ/n corresponds to the probability of getting heads, which can be interpreted as the rate of heads per toss. Since a toss happens every $1/n$ time units, we see that λ is the rate of heads per unit time. Or in other words, λ is the rate of service (per unit time). Alternatively, $1/\lambda$ is the average waiting time.

5. Prove the Borel-Cantelli lemma:

a) If A_n are events such that $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n$ happens for infinitely many $n) = 0$.

b) If A_n are independent events and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(A_n$ happens for infinitely many $n) = 1$.

c) Prove that if X_n are independent with $P(X_n = 1) = 1/n = 1 - P(X_n = 0)$, then $P(X_n \to 0) = 0$. Deduce that $X_n \to 0$ in probability, while $X_n \not\to 0$ almost surely.

6. Solve exercise 3.7, 3.11, 5.1, 5.3 from the textbook.

7. Let $f(x)$ be the pdf of the $t(2)$ distribution. Let $L(x) = \int_{-x}^{x} y^2 f(y)dy$. Prove that $L(x)$ is slowly varying.

8. Solve exercise 7.2 from the textbook. (What is meant there is to find a centering and scaling that would make that ratio converge weakly to a non-trivial limit.)