Solve all the underlined problems. The others are only recommended.

**Problem 1.1.** Let \( X_n \) be i.i.d. random variables with mean \( \mu \). Let \( S_n = X_1 + \cdots + X_n \). Prove that \( S_n - n\mu \) is a martingale in the natural filtration.

**Problem 1.2.** Let \( X_n \) be i.i.d. random variables with mean 0 and variance \( \sigma^2 \). Let \( S_n = X_1 + \cdots + X_n \). Prove that \( S_n^2 - n\sigma^2 \) is a martingale in the natural filtration.

**Problem 1.3.** Let \( X_n \) be i.i.d. random variables with \( P(X_1 = 1) = 1 - P(X_1 = -1) = p \in (0,1) \). Let \( S_n = X_1 + \cdots + X_n \). Prove that \( (p^{-1}(1-p))^S_n \) is a martingale in the natural filtration.

**Problem 1.4.** Prove that if \( T \) and \( S \) are stopping times then so are \( T \wedge S \), \( T \vee S \), and \( T + S \).

**Problem 1.5.** Assume the setting of Problem 1.3 and suppose \( p \neq 1/2 \).
(a) For \( x \in \mathbb{Z} \) let \( T_x = \min\{n \geq 0 : S_n = x\} \). Prove \( T_x \) is a stopping time.
(b) Let \( a \) and \( b \) be two positive integers. Prove that \( P(T_{-a} \wedge T_b < \infty) = 1 \).
(c) Use the martingale in Problem 1.2 to compute \( P(T_b < T_{-a}) \).
(d) Use the martingale in Problem 1.1 to compute \( E[T_{-a} \wedge T_b] \).

**Problem 1.6.** You have $3 in your pocket. You are given a coin that lands heads with probability \( p \). Every minute you toss the coin once. If it is heads you gain an extra $1. If it is tails you lose a $1. If you reach $0 then you will have $0 for the rest of the game, regardless of what your coin comes up. The (small) House has $5. So if you reach $8 you will have $8 for the rest of the game regardless of what your coin lands.

Is this a Markov chain? Why or why not? If it is, write its transition matrix.

**Problem 1.7.** You have a bag with 5 red balls and 3 blue ones. Every minute you draw a ball at random from the bag and throw it away. As soon
as the bag gets empty, your friend refills it with 5 red balls and 3 blue ones again, and you resume the game. Let $X_n$ denote the color of the ball you drew at time $n$. Is $X_n$ a Markov chain? Why or why not? If it is, write its transition matrix.

**Problem 1.8.** Consider a tiny web consisting of 6 sites. Site 1 is John’s homepage. John teaches calculus and his site points to sites 2 and 3. Site 2 is the course syllabus and does not point to any other sites. Site 3 is the calculus course website. It points back at John’s homepage and also at the course syllabus. It also points at Emily’s webpage, site 5. (Emily is the TA for the course.) Site 4 belongs to a friend of Emily’s, Jack. It points at Emily’s website and at Jack’s old website, site 6. Emily’s website points at both Jack’s pages, the new one 4 and the old one 6. Jack’s old webpage 6 points at his new webpage 4.

(a) Draw the graph corresponding to the above network.

(b) Consider the stochastic process where from each site you go equally likely to any of the sites it points to. What happens when you eventually reach site 2?

(c) Extend the stochastic process by the rule: from a site that does not point at any other site go equally likely to any site on the web (including the original site itself). Does this process give a Markov chain? Write its transition matrix. What are its communication classes? Which are transient and which are recurrent? Compute the invariant measure. Does it give positive probability to all sites? Can we use it to rank the transient states? Why or why not? (It is OK to use a computer or a laptop to compute the probability measure.)

(d) To overcome the problem with transient states, consider instead the following Markov chain: from a site that does not point to any other site, go equally likely to any site on the web (including the site itself); but from a site that does points somewhere, flip a coin that gives heads with probability 15%. If the coin lands heads, go equally likely to any site on the web (including the site itself). If, on the other hand, it lands tails then go equally likely to any site that the current site points to. Write the transition matrix for this chain. What are its communication classes? Which are transient and which are recurrent? Compute the invariant measure. Does it give positive probability to all sites?
(e) Use your results in part d) to rank the six sites.

You just worked out how Google ranks webpages!

**Problem 1.9.** Consider the Markov chain on nonnegative integers that does the following: from any site \( x \geq 0 \) it jumps to \( x + 1 \) with probability \( p \) and to 0 with probability \( 1 - p \). Is this chain transient? Null recurrent? Positive recurrent? You need to prove whatever you claim.

**Problem 1.10.** Consider a Markov chain with a finite state space \( S \). Suppose the Markov chain \( X_n \) starts at some state \( X_0 = x \). We want to compute the average number of visits \( N_y \) it makes to some state \( y \). If \( y \) is a recurrent state, then the number of visits is infinite. So the question makes sense only for states \( y \) that are transient and thus \( x \) has to also be a transient state. We would then be computing the number of visits to \( y \), before the Markov chain gets into some communicating class.

(a) Explain why for the purpose of the computation we want to perform, we can collapse each communication class into a single site. Let us denote these collapsed states by \( z_1, \ldots, z_r \) (where there are \( r \geq 1 \) of them). Let \( T = S \setminus \{x_1, \ldots, x_r\} \) denote the remaining (transient) sites. So \( x, y \in T \).

(b) Prove that \( E[N_y \mid X_0 = x] = \sum_{n=0}^{\infty} P(X_n = y \mid X_0 = x) \). (Hint: note that \( N_y = \sum_{n=0}^{\infty} 1\{X_n = y\} \).) Recall that \( P(X_n = y \mid X_0 = x) \) corresponds to the \((x, y)\)-entry in the matrix \( \Pi^n \) (where \( \Pi \) is the transition matrix). Hence, we have just shown that the average number of visits to \( y \), starting at \( x \), is the \((x, y)\)-entry of the matrix \( \sum_{n=0}^{\infty} \Pi^n \).

(c) Let \( Q \) denote the (symmetric) matrix of transitions from sites in \( T \) to sites in \( T \). Prove that \( Q^n \to 0 \). (Hint: what is the meaning of the entries of \( Q^n \?) Conclude that all the eigenvalues of \( Q \) are \(< 1\) and thus that \( I - Q \) is invertible.

(d) Prove that the \((x, y)\)-entry of \( \Pi^n \) is the same as the \((x, y)\)-entry of \( Q^n \). (Hint: if you arrange the rows and columns of \( \Pi \) so that first you have the recurrent sites then the transient ones you will notice that \( \Pi \) has on its diagonal an \( r \times r \) identity matrix then the matrix \( Q \), and above these two diagonal blocks it has a zero block.)

(e) Conclude that the desired average number of visits is given by the \((x, y)\)-entry of the matrix \((I - Q)^{-1}\).
**Problem 1.11.** Consider an irreducible Markov chain with a finite state space $S$. Suppose we want to compute the average number of steps it takes to go from a site $x$ to a site $y$.

(a) Explain why for the purpose of the computation we want to perform, we can make $y$ an absorbing site, i.e. we can assume $p(y, y) = 1$. Once we do that, all sites except $y$ become transient and $y$ becomes the only recurrent site.

(b) We can apply the method in Problem 1.10 to compute the average number of visits to any site $z \neq y$, starting at $x$. Explain how to use this to compute the average number of steps it takes to go from $x$ to $y$.

**Problem 1.12.** Consider the Markov chain on sites $\{0, 1, 2, 3, 4\}$ with transition matrix

$$
\Pi = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
$$

Sites 0 and 4 are absorbing and $\{1, 2, 3\}$ are transient.

(a) Compute the probabilities of absorption at 0 and at 4, starting at 1.

(b) Compute the average number of steps it takes to get absorbed, starting from 1.

(c) Write a computer code that simulates the Markov chain. Run 10,000 simulations of the chain starting at 1 and count the average number of times it ends up being absorbed at 0 versus 4. Also, for each of the simulations count the number of steps it took to get absorbed and count the average number of steps. Is it close to the computation you just did?
Problem 1.13. Consider the $3 \times 3$ board of Snakes and Ladders in the above figure. The rules are: every turn you toss a fair coin. Heads move you one step forward while tails move you two steps forward. If you are at the bottom of a ladder you move to its top right away. If you are at the mouth of a snake you slide down to its tail right away. You start at 1. The game ends when you reach 9.

(a) Calculate the average number of steps it takes one player to go from start to finish.

(b) Calculate the average number of steps it takes two players to go from start to finish. (Hint: the two players are playing independently, taking turns moving. Hence, we can consider a game where the two players make their moves simultaneously, but using two independent fair coins. The game ends when one of them reaches the finish. Thus, consider a Markov chain with state space being pairs of numbers. The chain starts at $(1,1)$. States $(a,b)$ with $a$ or $b$ being 9 are absorbing. Now, your task is to calculate the number of steps before absorption. The dimension of the matrix is rather large, and so once you set up the problem correctly you will need to use a computer to do the computation.)

(c) John and Mary are playing the game. John starts and then they take turns moving. What is the probability John wins? (Hint: If we instead make the two players move simultaneously, as in part (b), and John’s
position is represented by the first coordinate, then we are asking for
the probability the Markov chain is absorbed at a state of type \((9, b)\).
Collapse all these states into one and collapse the remaining ones [of type
\((a, 9)\) with \(a \neq 9\)] into one and answer the question.)

**Problem 1.14.** Consider a Markov chain with state space \(\mathbb{R}\) and tran-
sition kernel \(p(x, dy)\). Suppose \(p(x, dy) = p(x, y) dy\) for some positive mea-
surable function \(p(x, y)\). Suppose there exists an invariant measure \(\mu\).
Prove that \(\mu\) is absolutely continuous with respect to the Lebesgue measure. (So:
if the kernel has a probability density, then all invariant measures have prob-
ability densities.)

**Problem 1.15.** We want to show that the two-dimensional simple sym-
metric random walk is recurrent. Let \(|x|\) denote the Euclidean norm.

(a) Consider the function \(F(x) = \log |x|\), for \(x \neq 0\). Prove that \(\Pi F(x) - F(x) \leq C/|x|^4\) for some positive finite \(C\) and all large \(|x|\).

(b) Consider \(U(x) = 1/|x|\), for \(x \neq 0\). Prove that \(\Pi U(x) - U(x) \geq c/|x|^3\) for some positive finite \(c\) and all large \(|x|\).

(c) Fix any \(D > 0\) and consider the function \(V(x) = F(x) - DU(x)\), for \(x \neq 0\). Prove that \(\Pi V(x) \leq V(x)\) for \(|x|\) large enough. Note that \(V(x) \to \infty\) as \(|x| \to \infty\).

(d) Fix two positive numbers \(\ell < L\). Let \(x \in \mathbb{Z}^2\) be such that \(\ell < |x| < L\). Let \(T_L\) and \(T_\ell\) be, respectively, the times when the random walk exits the outer and the inner discs. Prove that

\[
P_x(T_L < T_\ell) \leq \frac{V(x)}{\inf_{|y| \geq L} V(y)}.
\]

(e) Prove that \(P_x(T_L < \infty) = 1\) for every \(L\). Conclude that \(P_x(T_\ell < \infty) = 1\), which proves recurrence.