Homework on the Ergodic Theorem

Exercise Erg.1. Let $\mathcal{A}$ be an algebra of subsets of some space Ω. Let $\mathcal{F}$ be the σ-algebra generated by $\mathcal{A}$. Let P be a probability measure on $(\Omega, \mathcal{F})$. Prove that for any $A \in \mathcal{F}$ and any $\varepsilon > 0$ there exists a $B \in \mathcal{A}$ such that $P(A \Delta B) \leq \varepsilon$.

In what follows, $(\Omega, \mathcal{F}, P)$ is a probability space and $T : \Omega \to \Omega$ is a bijection measurable function such that $P(A) = P(TA)$ for all $A \in \mathcal{F}$. T^{-1} is the inverse of T. T^0 is the identity map. For $n \in \mathbb{N}$, T^n denotes the composition of T with itself n times and T^{-n} is the composition of T^{-1} with itself n times.

Exercise Erg.2. Let $X_0 : \Omega \to \mathbb{R}$ be an L^1 random variable. For $n \in \mathbb{Z}$, let $X_n = X_0 \circ T^n$. Prove that $\{X_n : n \in \mathbb{Z}\}$ is a stationary sequence, i.e. for all $k, m \in \mathbb{N}$ and all $n_1 < n_2 < \ldots < n_k$ in $\mathbb{Z}$, the distribution of $\{X_{n_1+m}, \ldots, X_{n_k+m}\}$ is the same as that of $\{X_{n_1}, \ldots, X_{n_k}\}$.

Exercise Erg.3. Let $\mathcal{I} = \{A \in \mathcal{F} : A = T^{-1}A\}$. Prove that $\mathcal{I}$ is a σ-algebra. Prove also that for any $A \in \mathcal{F}$ with $P(A \Delta T^{-1}A) = 0$ there exists a $B \in \mathcal{I}$ such that $P(A \Delta B) = 0$.

Exercise Erg.4. Let $\mathcal{H}_0 = \{f \in L^2 : f = f \circ T \text{ almost surely}\}$. Prove that $\mathcal{H}_0^{\perp} = \{g \in L^2 : \forall f \in \mathcal{H}_0 \; E[fg] = 0\}$ is closed in L^2.

Exercise Erg.5. Let $A_n f = n^{-1} \sum_{k=0}^{n-1} f \circ T^k$. Suppose that for any $f \in L^2$ we have $\|A_n f - f\|_2 \to 0$. Prove that for any $p \geq 1$, for any $f \in L^p$, $\|A_n f - f\|_p \to 0$.