MATH 5050/6815: Homework 3 (Due Monday, March 17)

Problem 1 a) Use Wiener's Fourier series

$$W(t) = X_0 t + \frac{\sqrt{2}}{\pi} \sum_{j=1}^{\infty} \frac{X_j}{j} \sin(\pi j t) \quad \text{for } t \in [0,1],$$

to simulate standard Brownian motion on the time-interval $[0,1]$. Here, X_0, X_1, X_2, ... are i.i.d. standard normal random variables.

b) Use Donsker's theorem to simulate standard Brownian motion on the time-interval $[0,1]$ using a random walk. For the random walk increments use the following distributions:

i) ± 1 equally-likely. (Don't forget to scale by the standard deviation to get a standard Brownian motion.)

ii) Standard normal.

iii) Uniform(0,1). (Don't forget to center by the mean and scale by the standard deviation to get a standard Brownian motion.)

c) Do any of the Brownian motions you have generated above differ visually?

d) What is the variance of $W(1/2)$? Using each of the above codes (a), b.i), b.ii), and b.iii)), generate 1000 samples of $W(1/2)$ and compute the sample variance. Are they close to the actual value?

e) What is the correlation between $W(1/2)$ and $W(1) - W(1/2)$? Using b.i) generate 1000 pairs of samples of $W(1/2)$ and $W(1) - W(1/2)$ and compute the sample correlation coefficient. Is it close to your prediction?

f) Calculate the probability that the maximum of $W(t)$ over the interval $[0,1]$ is larger than 3. Using b.ii) generate 1000 samples of the path $\{W(t) : t \in [0,1]\}$ and for each sample compute the maximum of W over the interval $[0,1]$. Using these samples, estimate the probability the maximum is above 3. Compare with your theoretical value.

Problem 2 Assume $B(t)$ is a Brownian motion with diffusion coefficient σ^2. Let $a > 0$.

a) Show that $X(t) = B(at)$ is a Brownian motion with diffusion coefficient $a\sigma^2$.

b) Show that $Y(t) = aB(t)$ is a Brownian motion with diffusion coefficient $a^2\sigma^2$.

c) Show that $Z(t) = a^{-1/2}B(at)$ is a Brownain motion with diffusion coefficient σ^2.

Problem 3 Let $W(t)$ and $B(t)$ be independent standard Brownian motions.

a) Show that $X(t) = W(t) - B(t)$ is a Brownian motion. What is its diffusion coefficient?

b) True or False: With probability one, $W(t) = B(t)$ for infinitely many values of t.

Problem 4 Let $W(t)$ be standard Brownian motion. Compute the following probabilities. I want an exact answer, not an approximation. So you can use simulations to get an approximation of the probability and thus verify your answers, if you wish. But you cannot give them as THE answer. I want to see the calculus. You can of course round off the answer.

a) $P\{W(2) > 2\}$,

b) $P\{W(2) > W(1)\}$,

c) $P\{W(1) > W(2) > W(3)\}$,

d) $E[(W(4) - W(2))W(2)]$,

e) $E[W(2)^2]$,

f) $E[W(2)W(4)]$,

g) $P\{W(t) = 0 \text{ for some } t \text{ with } 2 \leq t \leq 3\}$,

h) $P\{W(t) < 4 \text{ for all } t \text{ with } 0 < t < 3\}$,

i) $P\{W(t) > 0 \text{ for all } t > 10\}$,

j) $P\{W(5) > 0 \,|\, W(2) > 0\}$.