Chapter 3

Continuous time Markov Chains.

The process \( \{X_t\} \) is indexed by \( \mathbb{R}_+ \), uncountably many r.v.'s involved.

The fact that time is continuous, allows us to model queuing systems. For example, let \( X_t \) denote

\[ X_t = \# \{ \text{people entering Smith's by time } t \} \]

How does this process behave?

\[ \begin{array}{c}
\text{3} \\
\text{2} \\
\text{1}
\end{array} \]

At some times (random) \( T_1, T_2, T_3, \ldots \), \( X_t \) decides to jump by 1, signifying that a customer entered the store.

\[ \begin{array}{c}
\text{Can two customers enter at exactly the same time?}
\end{array} \]

For reasonable assumptions:

1. Number of customers arriving during one time interval cannot affect the number of customers arriving during a different time interval.

2. The "average" rate at which customers arrive is constant.

3. Customers arrive one at a time.
Mathematical version of Assumptions.

Let \( 0 < s_1 \leq t_1 \leq s_2 \leq t_2 \leq \ldots \leq s_n \leq t_n \ldots \), \( n \in \mathbb{N} \).

Then the r.v.'s

\[ X_{t_1} - X_{s_1}, \, X_{t_2} - X_{s_2}, \ldots, \, X_{t_n} - X_{s_n} \]

are independent.

1. If the arrival rate is fixed and \( \lambda \), then in an interval of size \( t \), we expect \( \lambda t \) customers.

   In a small, tiny interval \((t, t+\Delta t)\), expect \( \lambda \Delta t \) customers.

Assume \( \lambda \Delta t \ll 1 \) (can be done if \( \Delta t \) is really small).

- \( \mathbb{P}\{X_{t+\Delta t} = X_t\} = \mathbb{P}\{\text{no customer arrived at } (t, t+\Delta t)\} = 1 - \lambda \Delta t + o(\Delta t) \)
- \( \mathbb{P}\{X_{t+\Delta t} = X_t + 1\} = \mathbb{P}\{1 \text{ customer arrived in } (t, t+\Delta t)\} = \lambda \Delta t + o(\Delta t) \)
- \( \mathbb{P}\{X_{t+\Delta t} = X_t + 2\} = o(\Delta t) \)

**Notation:** \( \lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0 \).

Stochastic process with \( X_0 = 0 \), \( X_t \) satisfying (1.2.3) is called Poisson process with parameter \( \lambda \).
Let $n$ sufficiently large (we'll decide later how large)

Write $X_t = \sum_{j=1}^{n} [X_{jt/n} - X_{(j-1)t/n}]$ (recall $X_0 = 0$)  

\[
\text{Independent (why?) (1)} \quad \text{Identically distributed (why?) (2)}
\]

\[
\sum_{j=1}^{n} \mathbb{P}\{X_{jt/n} - X_{(j-1)t/n} \geq 2 \} \geq 2 \text{ for some } j \leq n
\]

\[
\leq \sum_{j=1}^{n} \mathbb{P}\{X_{jt/n} - X_{(j-1)t/n} \geq 2 \} 
= n \cdot \mathbb{P}\{X_{t/n} \geq 2\} = o\left(\frac{t}{n}\right) \xrightarrow{n \to \infty} 0 \text{ as } n \text{ grows.}
\]

Then the i.i.d r.v. $X_{jt/n} - X_{(j-1)t/n}$ are 0,1 valued (Bemoulli) with prob. of success $\frac{t}{n}$ (why?) and their sum is Binomial $\text{Bin}\left(n, \frac{t}{n}\right) = Y$

\[
\mathbb{P}\{X_t = k\} \approx \sqrt{n} \mathbb{P}\{Y = k\} = \binom{n}{k} \left(\frac{t}{n}\right)^k \left(1 - \frac{t}{n}\right)^{n-k}
\]

\[
eq \lim_{n \to \infty} \mathbb{P}\{X_t = k\} = \frac{n!}{(n-k)!} \cdot \frac{1}{n^k} \cdot \frac{(at)^k}{k!} \cdot \left(1 - \frac{at}{n}\right)^{n-k}
\]

\[
= \frac{(at)^k}{k!} \lim_{n \to \infty} \frac{n!}{(n-k)! n^k} \cdot \frac{1}{n^k} \cdot \left(1 - \frac{t}{n}\right)^{n-k}
\]

\[
= e^{-at} \frac{(at)^k}{k!}
\]
\[ P \left( X_t = k \right) = e^{-\lambda t} \frac{(\lambda t)^k}{k!} \quad \text{Poisson}(\lambda) \text{ process} \]

Another way to view the Poisson process is with the event times \( T_i \).

**Waiting times**

Let \( T_n \) be the time between arrivals of \( n-1 \) and \( n \) customers.

Let \( Y_n = T_1 + T_2 + \cdots + T_n \) be the total waiting time to see \( n \) customers.

So,

\[
T_n = Y_n - Y_{n-1}
\]

**My computations**

\[
P \left( T_1 > t \right) = P \left( X_t = 0 \right) = e^{-\lambda t}.
\]

\[
T_1 \sim \text{Exp}(\lambda), \quad P \left( T_1 \leq s \right) = 1 - e^{-\lambda s}.
\]

**Memoryless property** of waiting times (Shift invariance of Poisson)

For \( i \geq 1 \),

\[
P \left( T_{i+1} > s + t \mid T_i > t \right) = P \left( X_{s+t} \leq i-1 \mid X_t \leq i-1 \right)
\]

\[
= P \left( X_s \leq i-1 \mid X_0 \leq i-1 \right)
\]

\[
= P \left( T_i > s \right).
\]
\[ P\{T_i > t\} = f(t). \]

Then \[ P\{T_i > s + t \mid T_i > t\} = P\{T_i > s\} \quad \iff \quad P\{T_i > s + t, T_i > t\} = P\{T_i > s\} \]

\[ P\{T_i > t\} \]

\[ f(s+t) = f(s)f(t) \]

\[ P\{T_i > t\} = e^{-\alpha t} \quad \text{for some } \alpha \]

\[ f(nt) = f(t)^n \quad f\left(\frac{t}{m}\right) = \frac{f(t)}{m} \]

\[ f(nt) = f(t)^n, \quad n \in \mathbb{N}_+ \]

\[ f(t) = f(\frac{t}{\alpha}) = e^{-\alpha t} \quad \exists \beta > 0 \quad f(\beta t) > 0 \quad \text{leftward } \Rightarrow f(t) = f(\frac{t}{\beta}) = e^{-\alpha t} \]

\[ \text{Done in Lec. 28} \]

1) \[ T = \min \{T_1, T_2, \ldots, T_n\} \quad T_i \text{'s are random alarm clocks, when will the first one go off?} \]

\[ P\{T > t\} = P\{\min \{T_1, \ldots, T_n\} > t\} = P\{T_1 > t, T_2 > t, \ldots, T_n > t\} = P\{T_1 > t\}P\{T_2 > t\} \cdots P\{T_n > t\} = e^{-bt_1}e^{-bt_2} \cdots e^{-bt_n} = e^{-(b_1 + \cdots + b_n)t}. \]
1) Recall that continuous density can be loosely interpreted:

\[ P\{T_i = t\} = b_i e^{-b_i t} dt \]

Then,

\[ P\{T = T_i \} = \int \cdots \int P\{T_i > t, \ldots, T_i = t, T_{i+1} > t, \ldots, T_n > t\} \]

\[ = \int_0^\infty \cdots \int_0^\infty e^{-b_1 t} e^{-b_2 t} \cdots e^{-b_{i-1} t} e^{-b_{i+1} t} \cdots e^{-b_n t} b_i e^{-b_i t} dt \]

\[ = \frac{b_i}{b_1 + \cdots + b_n} \]

The same result holds for infinite sequences as long as \( \sum b_i < \infty \).

(can also do it by verifying it for \( n=2 \), then using \( \cdots \))

Markov property for continuous chains!

Back to Lec 26

Let \( S \) be a finite state space. The process \( X_t \) is a Markov process on \( S \) iff for \( t > s \)

\[ P\{X_t = y \mid X_r, 0 \leq r \leq s\} = P\{X_t = y \mid X_s\} \]

Furthermore, \( X_t \) is time homogeneous

\[ P\{X_{t+s} = y \mid X_s = x\} = P\{X_t = y \mid X_s = x\} \]
Instruction of a continuous time M.P.

- Define rates \( \alpha(x,y) \geq 0 \) for each pair \((x,y) \in S^2, x \neq y\).

So, if we are on state \( x \), we jump to a state \( y \) with rate \( \alpha(x,y) \) (see this as a certain proportion of time where you decide where to jump).

Define \[ \alpha(x) = \sum_{y \neq x} \alpha(x,y) \] to be the total rate at which the chain is changing from state \( x \).

A stochastic process that is time-homogeneous \& Markov in \( S \) satisfies:

**Assumption 1:** \[ P\{ X_{t+\Delta t} = z \mid X_t = x \} = 1 - \alpha(z) \Delta t + o(\Delta t) \]

**Assumption 2:** \[ P\{ X_{t+\Delta t} = y \mid X_t = x \} = \alpha(x,y) \Delta t + o(\Delta t) \]

Heuristically:

On each site \( y \in S \) attach independent exponential alarm clocks with rates \( \alpha(x,y) \), for all \( x \in S \).

Then the chain stays in state \( x \) until one of those alarms goes off and then, we jump to state \( y \).

- This is Markovian—only the alarms involving the current state are important.
- This is time homogeneous (why?).

\[ \begin{array}{ccc}
\alpha_{01} & 1 & T_{0,1} \sim \exp(\lambda_{01}) \\
2 & T_{0,2} \sim \exp(\lambda_{02}) \\
3 & T_{0,3} \sim \exp(\lambda_{03})
\end{array} \]

Say time now is 23.5. Say \( T_{0,1} = 0.7 \), \( T_{0,2} = 5.1 \), \( T_{0,3} = 7.3 \).

The next jump is then to 2 at time \( 23.5 + 5.1 = 28.6 \).
Define: \( p_x(t) = \mathbb{P}\{X_t = x\} \) and use the limit definition of derivative,

\[
p_x'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left( \mathbb{P}\{X_{t+\Delta t} = x^2\} - \mathbb{P}\{X_t = x^2\} \right)
\]

Now, \( \mathbb{P}\{X_{t+\Delta t} = x^2\} = \sum_{y \in S} \mathbb{P}\{X_{t+\Delta t} = x^2 | X_t = y\} \mathbb{P}\{X_t = y\} \)

\[
= \mathbb{P}\{X_{t+\Delta t} = x^2 | X_t = x\} \mathbb{P}\{X_t = x\} + \sum_{y \neq x} \mathbb{P}\{X_{t+\Delta t} = x^2 | X_t = y\} \mathbb{P}\{X_t = y\}
\]

\[
\Rightarrow \quad p_x'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left( -\mathbb{P}\{X_t = x\} (1 - \mathbb{P}\{X_{t+\Delta t} = x | X_t = x\}) + \sum_{y \neq x} \mathbb{P}\{X_{t+\Delta t} = x^2 | X_t = y\} \mathbb{P}\{X_t = y\} \right)
\]

\[
= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left( -\alpha(x) p_x(t) \Delta t + \sum_{y \neq x} \alpha(y,x) p_y(t) \Delta t \right) + \sum_{y \neq x} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \alpha(y,x) p_y(t) \Delta t + 0(\Delta t)
\]

\[
p_x'(t) = -\alpha(x) p_x(t) + \sum_{y \neq x} \alpha(y,x) p_y(t)
\]

This system of differential equations is called the infinitesimal generator!

In (x) can be written as \( \bar{p}'(t) = \bar{p}(t) \cdot A \).
A has row sum = 0
non-negative diagonal entries
non-negative non-diagonal entries
Always has a 0 eigenvalue!

In terms of transition matrices,

\[ P_t(x,y) = \mathbb{P}\{X_t = y \mid X_0 = x\} \]

\[ P_t = \begin{pmatrix} P_t(x,y) \end{pmatrix}, \quad x, y \in S. \]

Then, the differential equation can be written as

\[ \frac{d}{dt} P_t = P_t A, \quad P_0 = I \]

Then

\[ P_t = e^{tA} \quad \text{(matrix exponential).} \]

Definition:

\[ e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \]

Also:

\[ A = \lim_{t \to \infty} \frac{P_t - I}{t} \]

(So \( \pi \) inv. meas. \( \pi A = 0 \))

→ Homework problem (will be assigned): If \( A = QDQ^{-1} \) with \( D \) diagonal, show that \( e^{tA} = Q e^{tD} Q^{-1} \).

Does \( \ast \) always converge? Yes, all eigenvalues have either 0 or have non-positive real part!
What is \( tA \)?

\[
\lim_{t \to \infty} tA = \begin{bmatrix} \frac{2 \sqrt{3}}{3} & \frac{1}{3} \\
-\frac{\sqrt{3}}{3} & \frac{2}{3} \end{bmatrix}
\]

\[
T = \begin{bmatrix} \frac{2 \sqrt{3}}{3} & \frac{1}{3} \\
-\frac{\sqrt{3}}{3} & \frac{2}{3} \end{bmatrix}
\]

\[
\mathbf{q} = \begin{bmatrix} 0 \\
\frac{\sqrt{3}}{2} \end{bmatrix}
\]

\[
\mathbf{p} = \begin{bmatrix} 1 \\
0 \end{bmatrix}
\]

\[
A = \begin{bmatrix} 1 & -3 \\
3 & -2 \end{bmatrix}
\]

\[
(tA)^n = \begin{bmatrix} \frac{2 \sqrt{3}}{3} & \frac{1}{3} \\
-\frac{\sqrt{3}}{3} & \frac{2}{3} \end{bmatrix}^n
\]

\[
\text{Diagonalize: }
\]

\[
A = \begin{bmatrix} 1 & -3 \\
3 & -2 \end{bmatrix}
\]

\[
det(A - \lambda I) = 0 \Rightarrow \lambda^2 + 3\lambda - 12 = 0 \Rightarrow \lambda = 2, -3
\]

\[
A^n = \begin{bmatrix} 2^n & 3(n-1)2^{-n} \\
3^{n-1} & 2^n \end{bmatrix}
\]

\[
A^{-1} = \begin{bmatrix} -1 & 1 \\
1 & -1 \end{bmatrix}
\]

\[
A^0 = I
\]

Irreducible 2.

State continuity summarize: axillary's hands add up to 0.
first time a clock rings is again an exp.

random variable y rate exactly \( x(y) = \sum_{\text{neighbors}} x(x,y) \)
and probability the \( (x,y) \) bond rings is \( \frac{\sigma(x,y)}{\sigma(x)} \).

So we can replace the original MC w/ an equivalent one
that only has clocks on the sites. Site \( x \) has an exp clock
with rate \( x(x) \). When the clock Rings, it's time for a jump.
MC jumps to \( y \) w/ prob. \( \frac{\sigma(y)}{\sigma(x)} \).

This can even be done for a countable state space, as long as
\( \sigma(x) < \infty \).

If \( \alpha = \sup_{x} \sigma(x) < \infty \) (always the case for a finite space), then
we can even use one clock for all sites! The rate of this
clock is \( \alpha \). When it rings, the chain moves, but now the moves
are not necessarily to \( y \neq x \). The chain could stay put at \( x \)
and wait for the next ring. This is b/c if the rates \( x \) and \( x(y) \) differ, then we cannot use the same rate \( x \) \( x(y) \)
unless we allow for staying @ the same with the lower rate.

This will be the subject of a HW exercise.
$X_t$ is irreducible, continuous time MC, finite state space $S$

Define $T = \inf \{ t : X_t = z \}$

$b(x) = \mathbb{E}(T | X_0 = x)$

Then $b(z) = 0$ (Right?)

Define $S$ to be the first time that the chain changes state

\[
\mathbb{E}(T | X_0 = x) = \mathbb{E}(S | X_0 = x) + \sum_{y \in S} \mathbb{P}(X_S = y | X_0 = x) \mathbb{E}(T | X_0 = y)
\]

First note that $Y \geq T$ (why?) We can rewrite

\[
y = T + \inf \{ t > T : X_t = z \}
\]

\[
y = T + \inf \{ t > 0 : X_{T+t} = z \}
\]

\[
y = T + \inf \{ t > 0 : X_{T+t} = z \} \mathbb{1}_{\{ X_T = y \}}
\]

Then

\[
\mathbb{E}(T | X_0 = x) = \mathbb{E}(S | X_0 = x) + \sum_{y \in S} \mathbb{P}(X_S = y | X_0 = x) \mathbb{E}(T | X_0 = y)
\]

\[
\mathbb{E}(T | X_0 = x) = \mathbb{E}(T | X_0 = x) + \sum_{y \in S} \mathbb{P}(X_T = y | X_0 = x) \mathbb{E}(T | X_0 = y)
\]

\[
\mathbb{E}(T | X_0 = x) = \mathbb{E}(T | X_0 = x) + \sum_{y \in S} \mathbb{P}(X_T = y | X_0 = x) \mathbb{E}(Y | X_0 = y)
\]
Now, $T$ is exponential with rate $\alpha(x) \Rightarrow$

$$E(S | X_0 = x) = \frac{1}{\alpha(x)}$$

$$P(S = y | X_0 = x) = \begin{cases} \text{the first alarm that rang with } y \text{'s} & | X_0 = x \\ \frac{\alpha(x,y)}{\alpha(x)} & (why?) \end{cases}$$

$b(\tilde{x}) = 0$

so the sum becomes

$$b(x) = \frac{1}{\alpha(x)} + \sum_{y \neq x, \tilde{x}} \frac{\alpha(x,y)}{\alpha(x)} \cdot b(y)$$

$$= \alpha(x) \cdot b(x) = 1 + \sum_{y \neq x, \tilde{x}} \frac{\alpha(x,y)}{\alpha(x)} \cdot b(y)$$

$$0 = 1 + \sum_{y \neq x, \tilde{x}} \alpha(x,y) \cdot b(y) - \alpha(x) \cdot b(x)$$

matrix form:

\[
\begin{bmatrix}
0 \\
\end{bmatrix} = \begin{bmatrix}
1 \\
\end{bmatrix} + \begin{bmatrix}
\tilde{\Lambda} \\
\end{bmatrix} \begin{bmatrix}
b \\
\end{bmatrix}
\]

where $\tilde{\Lambda}$ is obtained by

$$A$$ by deleting the row and column associated to $\tilde{x}$.

Remember $z$ is specified from the problem. Different $a$'s mean different $\tilde{\Lambda}$ so the process must be repeated.
Example 2: 

\[ A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix} \]

\[ A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ A = \begin{bmatrix} -4 & 3 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix} \]

Common rate: 2/4

\[ P = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} \]

\[ \pi P = \pi \Rightarrow \pi A = \pi \]

\[ \pi = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \]

\[ P \text{ has only } \pi \text{ e-vectors} \]

\[ \text{time } t \rightarrow \pi = \pi A^t \]

\[ \pi = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \]

\[ \pi A = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \]

\[ \pi \text{ is the stationary state} \]

\[ Q = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} \]

\[ (I - Q) = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} \]

\[ \pi A = 0 \Rightarrow \pi A = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \]

\[ e^{-3t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} \]

\[ e^{-4t} = \begin{bmatrix} 1/12 & -1/4 & 1/12 \\ -1/12 & 1/4 & -1/12 \\ 1/12 & -1/4 & 1/12 \\ -1/12 & 1/4 & -1/12 \end{bmatrix} \]

\[ \pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/3 & 2/3 \\ 0 & 1/3 & 2/3 \end{bmatrix} \]

\[ \text{Compute the expected value of the time needed to get from } \]

\[ \text{0 to 3.} \]

\[ \text{So } 2 = 3! \text{ We want } b(0) ! \]

\[ \tilde{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow [-\tilde{A}]^{-1} = \begin{bmatrix} 5/3 & 2/3 & 1/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix} \]

\[ \text{So } \frac{1.6}{4} = 0.4 \text{ time units.} \]
\[ b = \begin{bmatrix} -A \end{bmatrix}^{-1} \begin{bmatrix} 1 \end{bmatrix} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \]

\[ b(0) = \frac{1}{3}, \quad b(1) = \frac{1}{3}, \quad b(2) = \frac{1}{3} \]

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**Birth & Death Chain**

**Lec. 20**

For this section: Small \( n \) denotes population.

- Birth rules: \( \lambda_n, \ n = 0, 1, 2, \ldots \)
- Death rules: \( \mu_n, \ n = 0, 1, \ldots \)

\( \mu_0 = 0 \). (why?)

Standing assumptions (changes of scale): \( n \rightarrow n+1 \) or \( n \rightarrow n-1 \)

- Transitions

\[ P\{ X_{t+\Delta t} = n+1 | X_t = n \} = \lambda_n \Delta t + o(\Delta t) \]
\[ P\{ X_{t+\Delta t} = n-1 | X_t = n \} = \mu_n \Delta t + o(\Delta t) \]
\[ P\{ X_{t+\Delta t} = n | X_t = n \} = 1 - (\mu_n + \lambda_n) \Delta t + o(\Delta t) \]

In a time interval, either a birth occurs (only) or a death occurs (only) or nothing at all!

Defining

\[ P_n(t) = P\{ X_t = n \} \]

we have a system of

\[ P'_n(t) = \mu_{n+1} P_{n+1}(t) + \lambda_n P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) \]

(Try to prove this!)
Examples

Poisson process with rate parameter $\lambda$ is birth and death with

$\lambda_n = \lambda$, $\gamma_n = 0$ (How to prove this?) Check equations!

Population models

Assume $n$ is the number of individuals at a given time.

Each individual reproduces at rate $\lambda$ and dies with rate $\gamma$.

Then $\lambda_n = n\lambda$, $\gamma_n = \gamma n$.

With immigration: $\lambda_n = n\lambda + \nu$, $\gamma_n = \gamma n$.

Fast growing models: pure birth!

Defining $Y_n = \inf \{ t : X_t = n \}$ and let $Y_\infty = \tau_1 + \tau_2 + \ldots$ be the time where the population becomes infinite. $T_i$ is time of $i$th auto-germ birth.

Then $E(Y_\infty) = \sum_{i=1}^{\infty} E(T_i) = \sum_{i=1}^{\infty} \frac{1}{i\lambda} < \infty$.

$\Rightarrow Y_\infty < \infty$ with probability 1.

$\Rightarrow$ Explosion!

It always occurs when $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$ (iff statement).
Discrete time MC that mimics the continuous one when it moves:

\[ p(n, n+1) = \frac{\mu_n}{\mu_n + \lambda_n} \quad p(n, n+1) = \frac{\lambda_n}{\mu_n + \lambda_n} \]

**Lemma** Continuity time B2D is recurrent iff discrete B2D is.

**Thm** B2D are transient iff \( \sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \infty \). (*)

**Example:**

Using models:

Service rate = \( \mu \), arrival rate = \( \lambda \), 1 server M/M/1

\[ \Rightarrow \lambda_n = \lambda, \mu_n = \mu \]

M/M/k: Service rate \( \mu_n = \mu, n \leq k \) \quad \lambda_n = \frac{\lambda}{k} \quad n > k

M/M/\infty: Service rate \( \mu_n = \mu \), arrival \( \lambda_n = \lambda \)

Then apply (*) to decide recurrence and transience.

M/M/1 \( \sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \sum_{n=1}^{\infty} \left( \frac{\mu}{\lambda} \right)^n < \infty \) if \( \mu < \lambda \) = Transience

M/M/1 \( \sum_{n=1}^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \sum_{n=1}^{\infty} \left( \frac{\mu}{\lambda} \right)^n < \infty \) if \( \mu > \lambda \Rightarrow \text{recurrence.} \)

M/M/k For all \( n > k \) \( \mu_1 \cdots \mu_n \ Quad \frac{k!}{\lambda_1 \cdots \lambda_n} \frac{(ky)^n}{k^n} \quad < \infty \) if \( ky < \lambda \)
variant distribution

\[ \sum_{n=0}^{\infty} \pi_n \frac{\gamma_n}{\lambda_1 \cdots \lambda_n} = \sum_{n=1}^{\infty} n \cdot \left( \frac{\gamma_n}{\lambda_n} \right)^n = 0 \quad \text{always!} \quad (\text{90}) \]

Lec. 3

assume irreducible chains!

inverse distribution

\[ \lim_{t \to \infty} P_t \{ X_t = n \mid X_0 = m \} = \pi(n) \]

\[ \pi \cdot P_t = \pi \]

(i would be infinite)

one exists,

\[ \pi \cdot P_t = \pi = 0 \quad \Rightarrow \quad \pi \cdot A \cdot P_t = 0 \]

\[ (\Rightarrow \pi A = 0) \]

assume \( P_t(n) = \pi(n) \)

then, from the diff eq's of BTD chains,

\[ 0 = \lambda_{n-1} \pi(n-1) + \gamma_{n+1} \pi(n+1) - (\lambda_n + \gamma_n) \pi(n) \]

For \( n = 0 \), \( \lambda_{-1} \pi(0) = 0 \), \( \Rightarrow \pi(1) = \frac{\lambda_0}{\gamma_1} \pi(0) \)

or \( n \geq 1 \), \( \pi(n) = \frac{\gamma_{n+1} \pi(n+1) - \lambda_n \pi(n)}{\lambda_{n+1} \pi(n+1) - \lambda_n \pi(n)} = \frac{\lambda_{n+1} \pi(n) - \lambda_n \pi(n)}{\lambda_{n+1} \pi(n) - \lambda_n \pi(n)} \]

\[ \Rightarrow \pi(n+1) = \frac{\lambda_{n+1}}{\gamma_{n+1}} \pi(n) \quad \Rightarrow \quad \pi(n) = \frac{\lambda_n \pi(n-1)}{\gamma_{n+1} \pi(n)} \]

\[ \Rightarrow \pi(n) = \frac{\lambda_0 \cdots \lambda_{n-1}}{\gamma_1 \cdots \gamma_n} \pi(0) \]
Then $\pi$ can be made into a probability measure iff

$$
\sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} < \infty.
$$

and

$$
\pi(n) = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \cdot \frac{1}{q^n}
$$

So the chain is positive recurrent iff the sum converges.

**Example**

$\text{1/M/1 queue}$

$$
\sum_{n=0}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \sum_{n=0}^{\infty} \left( \frac{\lambda}{\mu} \right)^n = \frac{1 - \frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} = 1
$$

if $\lambda < \mu$.

$$
\pi(n) = \left( 1 - \frac{\lambda}{\mu} \right) \left( \frac{\lambda}{\mu} \right)^n
$$

Expected queue length in equilibrium:

$$
\sum_{n=0}^{\infty} n \pi(n) = \sum_{n=0}^{\infty} n \left( 1 - \frac{\lambda}{\mu} \right) \left( \frac{\lambda}{\mu} \right)^n = \frac{\lambda}{\mu} \left( 1 - \frac{\lambda}{\mu} \right)^2 = \frac{\lambda}{\mu - \lambda}.
$$

Class do $M/M/\infty$ !
Population model: $\lambda_n = n\lambda, \mu_n = n\mu$

It's a branching process in count time!

When an individual is born, it does not die out in the next generation. It lives for a random time that is a $\exp(\mu)$. While it lives, it keeps giving offsprings every $\exp(\lambda)$ time. So, on average, during its life time of $\mu$, it will give $\lambda/\mu$ offsprings. I.e. what was $\mu$ for us in the discrete model is now $\frac{\lambda}{\mu}$. We expect exponential growth if $\frac{\lambda}{\mu} > 1$ and extinction if $\frac{\lambda}{\mu} \leq 1$. We expect exponential extinction if $\frac{\lambda}{\mu} < 1$.

Lec. 32

Let's see.

Transience if $\sum_{n=1}^{\infty} \frac{n!\mu^n}{\lambda^n} < \infty$ so if $\mu < \lambda$.

Positive rec. if $\sum_{n=1}^{\infty} \frac{(n-1)!\mu^n}{\lambda^n} < \infty$ so if $\mu > \lambda$. (To continue the chain after it reaches 0, we can say $\lambda = \lambda$ instead of $\lambda = \lambda$.)

$\lambda = \mu$: null rec. (we didn't prove this in the discrete case, but now we did!)

Let $E(t) = E[\sum_{i=0}^{\infty} \xi_i | X_0 = 1] = \sum_{n=1}^{\infty} nP(X_0 = n | X_0 = 1)$ (assuming here $\lambda_0 = 0$)

Recall: $\frac{1}{\lambda} P_t = PA$. 

\[ \Rightarrow \sum_{n=1}^{\infty} nP_t(1/n) \]
\[
\begin{align*}
S_0 \ f(t) = & \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} p_k(1,k) A(k,n) = \sum_{n=1}^{\infty} \left( (n-1) \lambda p_t(1,n-1) + (n+1) \mu p_t(1,n+1) - \lambda \sum_{n=1}^{\infty} (\lambda + \mu)^{n-2} p_t(1,n) \right) \\
= & \sum_{n=2}^{\infty} n(n-1) p_t(1,n-1) + \mu \sum_{n=0}^{\infty} (n+1) p_t(1,n+1) - \lambda \mu \sum_{n=1}^{\infty} n^2 p_t(1,n) \\
= & \sum_{n=1}^{\infty} \left( (\lambda + \mu) n + \mu (n-1) - (\lambda + \mu)^2 \right) p_t(1,n) \\
= & (\lambda - \mu) \sum_{n=1}^{\infty} n p_t(1,n) = (\lambda - \mu) f(t)
\end{align*}
\]

\[
f(0) = 1 \quad (\text{why?})
\]

\[
S_0 \ f(t) = e^{(\lambda - \mu) t}
\]

\(\lambda > \mu \implies \text{exp. growth}\)

\(\lambda < \mu \implies \text{exp. decay}\)

\(\lambda = \mu \implies \text{Population remains constant, on average} \)

but extinction does happen 100% of the time!

Explanation: Null recurrence

(population has a good chance of getting huge before eventually getting extinct.)
Poisson processes $\lambda_n = \lambda$, $\mu_n = 0 \forall n \geq 0$.

We want $P(X_t = n | X_0 = 0) = p_n(t)$ Call this $g_n(t)$

$$g_0'(t) = -\lambda g_0(t) \quad \text{(using } \frac{d}{dt}P_t = P_t A \text{ & that nothing goes to 0!)}$$

$g_0(0) = 1$. So $g_0(t) = e^{-\lambda t}$.

For $n \geq 1$:

$$g_n(t) = \lambda g_{n-1}(t) - \lambda g_n(t)$$

or the rate from $m$ to $n$.

Let $f_n(t) = e^{\lambda t} g_n(t)$.

$$f_n'(t) = e^{\lambda t} g_n'(t) + \lambda e^{\lambda t} g_n(t) = \lambda e^{\lambda t} f_{n-1}(t) - \lambda e^{\lambda t} f_n(t) + \lambda e^{\lambda t} g_n(t)$$

So $f_n'(t) = \lambda f_{n-1}(t)$ and $f_n(0) = 0$.

$$f_n''(t) = \lambda f_{n-1}'(t) = \lambda^2 f_{n-2}(t) \quad \text{Lec. 33}$$

$$f^{(m)}_n(t) = \lambda^m f_{n-m}(t) \quad 0 \leq m \leq n.$$ 

We see that $f_n^{(m)}(0) = 0$ for $0 \leq m < n-1$ and $n \geq 1$.

So $f_n^{(0)}(t) = \lambda^0 f_0(t) = 1$, $f_n^{(1)}(t) = \lambda t$, $f_n^{(2)}(t) = \frac{\lambda^2 t^2}{2}$

$$\ldots f_n(t) = \frac{\lambda^m t^m}{m!} \quad \text{and } g_n(t) = e^{-\lambda t} \frac{\lambda^t}{n!} \quad n \geq 1$$

$n = \infty$ same formula.

This shows that $X_t \sim \text{Poisson}(\lambda t)$

After 2 hours we got fewer than 2 jumps

$$\mathbb{E} X_t \leq 1|X_0 = 0 = P(X_2 = 1|X_0 = 0) + P(X_2 = 0|X_0 = 0) = \lambda t e^{\lambda t} + e^{\lambda t}.$$

$$P(N = 5, N_2 = 6 | N_2 = 6) = \frac{P(X = 5, X_2 = 6)}{P(X_2 = 6)} = \frac{\lambda^5 e^{\lambda t}}{e^{\lambda t} \frac{\lambda^2 t^2}{2}} = \frac{6!}{5!} \frac{e^{\lambda t} \lambda^5 t^5}{e^{\lambda t} \lambda^2 t^2}$$

given 6 jumps happened in the first 2 hours, what is the prob 5 of them happened in the first hour? Binomial!