Lec 37

Multi-species?

Discrete time:

$X_n = (X_{1,n}, \dots, X_{m,n})$ m types

$$\sum_{l_1,\dots,l_m=0}^{\infty} P_k(l_1, \dots, l_m) = 1 \qquad k=1,\dots,m$$

prob a type k parent gives l_1 children of type 1 ... l_m children of type m

$$X_{k,n} = \sum_{i=1}^{m} \sum_{j=1}^{X_{i,n-1}} Z_{i,k,j,n-1}$$

$X_{i,n-1}$: # of individuals of type i in gen. n−1

$X_{k,n}$: # individuals of type k in gen. n

$Z_{i,k,j,n-1}$: # of type k offsprings of individual #j of type i in gen. n−1

Tool: $\displaystyle \phi_k(s_1, \dots, s_m) = \sum_{l_1,\dots,l_m=0}^{\infty} P_k(l_1 \dots l_m)\, s_1^{l_1} \cdots s_m^{l_m}$

multivariate prob. gen. function of type k species

Is $P(\exists n : \underbrace{X_{1,n} = \dots = X_{m,n} = 0}_{\text{total extinction}}) = 1$ or < 1?

$$a_{k,j} = \sum_{l_1,\dots,l_m=0}^{\infty} l_j P_k(l_1, \dots, l_m)$$

= avg. # of offsprings of type j from a parent of type k

Let A be the $m \times m$ matrix w/ entries a_{kj}.
Then:

<u>Th.</u> A has a positive e-val. μ
s.t. all other e-values λ have $|\lambda| < \mu$.

And $\mu > 1 \Rightarrow$ survival
$\mu \leq 1 \Rightarrow$ extinction

For prob. of survival:
solve $S_k = \phi_k(S_1, \ldots, S_m)$ $i = 1, \ldots, m$

(m nonlinear equations w/ m variables)

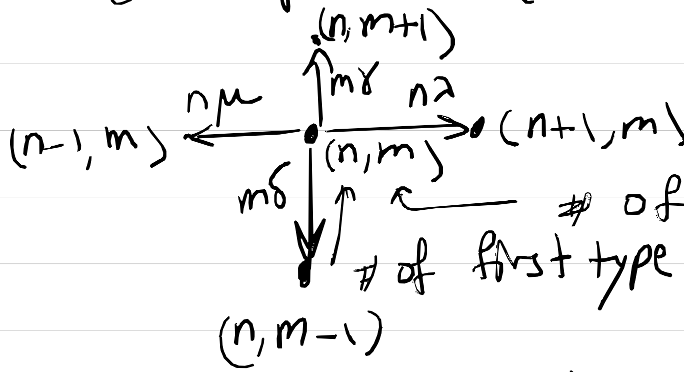
S_k gives the probability of extinction of species #k

- When there is total extinction, i.e. $\mu \leq 1$, $S_1 = \cdots = S_m = 1$ is the <u>only</u> solution.

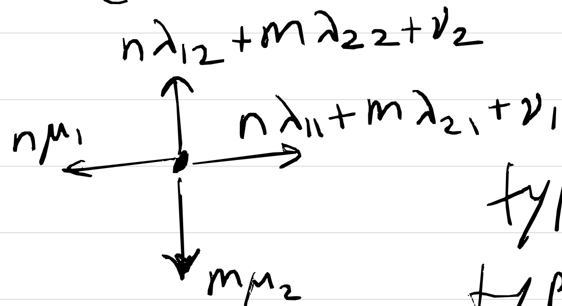
- When there isn't total extinction (i.e. $\mu > 1$), some species may survive and some others may not. So some S_k's may $= 1$ and some S_k's may be < 1. (At least one S_k will be < 1)

Cont. time?

two species (for simplicity)



two indep. branching processes



type 1 dies at rate μ_1

type 2 dies at rate μ_2

no killing

while alive, type 1 → type 1 rate λ_{11}

→ type 2 rate λ_{12}

type 2 → type 1 rate λ_{21}

→ type 2 rate λ_{12}

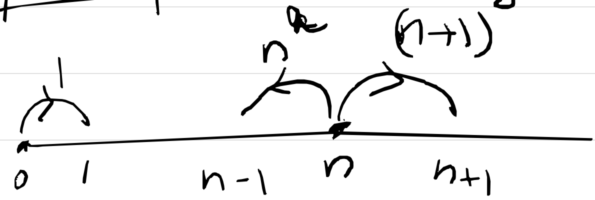
type 1 immigration at rate ν_1

type 2 immigration at rate ν_2

(could also add migrations)

Examples of birth-death processes:

$a, b > 0$



transience vs rec.

$$\sum_{n=0}^{\infty} \prod \frac{\mu's}{\lambda's} = \sum_{n=0}^{\infty} \frac{(n!)^a}{((n+1)!)^b} \quad \begin{cases} = \infty & \text{if } a > b \quad \text{rec.} \\ < \infty & \text{if } a < b \quad \text{trans.} \end{cases}$$

$a = b$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)^a} \quad \begin{cases} = \infty & \text{if } a = b \leq 1 \quad \text{rec.} \\ < \infty & \text{if } a = b > 1 \quad \text{trans.} \end{cases}$$

pos. vs null rec.

$$\sum_{n=0}^{\infty} \prod \frac{\lambda's}{\mu's} = \sum_{n=0}^{\infty} \frac{(n!)^b \leftarrow \lambda_0 \cdots \lambda_{n-1}}{(n!)^a \leftarrow \mu_1 \cdots \mu_n}$$

$$\begin{cases} = \infty & \text{if } a \leq b \quad \text{not pos. rec} \\ < \infty & \text{if } a > b \quad \text{pos. rec.} \end{cases}$$

So:

transience if $0 < a < b$ or $a = b > 1$

pos. rec. if $a > b > 0$

null rec. if $0 < a = b \leq 1$

Explosion: $n \xrightarrow{\lambda_n} n+1$ $\quad \mu_n = 0$

how fast (on average) do we get to ∞?

$n \to n+1$ takes (on average) $\frac{1}{\lambda_n}$ time

So $0 \longrightarrow \infty$: $\sum_{n=0}^{\infty} \frac{1}{\lambda_n}$

Examples:

Poi(λ) $\quad \sum \frac{1}{\lambda} = \infty$: takes (on average) ∞ time to get to ∞

$n \xrightarrow{n} n+1 \quad \sum \frac{1}{n} = \infty$: also, ∞ time to ∞

$n \xrightarrow{n^2} n+1 \quad \sum \frac{1}{n^2} < \infty$! gets to ∞ in finite time! (Explosion)

Practice:

$$\begin{cases} \lambda_n = 1 + \frac{1}{n+1}, & n \geq 0 \\ \mu_n = 1 \quad n \geq 1, & \mu_0 = 0 \end{cases}$$ trans.? null rec.? pos. rec.?

$$\begin{cases} \lambda_n = 1 - \frac{1}{n+2}, & n \geq 0 \\ \mu_n = 1, \; n \geq 1, & \mu_0 = 0 \end{cases}$$ trans.? null rec.? pos. rec.?

— The End —