Lec 36

individual lives for $\exp(\mu)$ amount of time. On avg. $\frac{1}{\mu}$

gives birth every $\exp(\lambda)$ time. ← rate

So in $\frac{1}{\mu}$ time units, gives $\frac{\lambda}{\mu}$

$\lambda > \mu \Rightarrow$ exp. growth ← transient

$\lambda < \mu \Rightarrow$ exp. decay ← pos. rec.

$\lambda = \mu \Rightarrow$ balance and we found it is null rec.

so still extinction.

$\frac{\lambda}{\mu}$ plays the role of the mean # of descendents

Branching w/ immigration and migration:

$$\sum_{n=0}^{\infty} \frac{(\mu+\gamma)(2\mu+\gamma)\cdots(n\mu+\gamma)}{(\lambda+\nu)(2\lambda+\nu)\cdots(n\lambda+\nu)}$$

$$= \sum_{n=0}^{\infty} \frac{(1+\frac{\gamma}{\mu})(1+\frac{\gamma}{2\mu})\cdots(1+\frac{\gamma}{n\mu})\,\mu^n}{(1+\frac{\nu}{\lambda})(1+\frac{\nu}{2\lambda})\cdots(1+\frac{\nu}{n\lambda})\,\lambda^n}$$

$$\left(1+\frac{\gamma}{\mu}\right)\cdots\left(1+\frac{\gamma}{n\mu}\right) = e^{\log(1+\frac{\gamma}{\mu})+\cdots+\log(1+\frac{\gamma}{n\mu})}$$

$$\log\left(1+\frac{\gamma}{n\mu}\right) \sim \frac{\gamma}{n\mu}$$

So $$\sum_{k=1}^{n} \log\left(1+\frac{\gamma}{k\mu}\right) \sim \frac{\gamma}{\mu}\sum_{k=1}^{n}\frac{1}{k} \sim \frac{\gamma}{\mu}\int_1^n \frac{1}{x}\,dx$$
$$= \frac{\gamma}{\mu}\log n$$

So $$\left(1+\frac{\gamma}{\mu}\right)\cdots\left(1+\frac{\gamma}{n\mu}\right) \sim e^{\frac{\gamma}{\mu}\log n} = n^{\gamma/\mu}$$

So $$\sum \prod \frac{\mu's}{\lambda's} \approx \sum \frac{n^{\gamma/\mu}}{n^{\nu/\lambda}}\left(\frac{\mu}{\lambda}\right)^n$$
$$= \sum n^{\frac{\gamma}{\mu}-\frac{\nu}{\lambda}}\left(\frac{\mu}{\lambda}\right)^n$$

$$\boxed{\mu < \lambda \Rightarrow \sum \prod \frac{\mu's}{\lambda's} < \infty \text{ so trans.}}$$
$$\mu > \lambda \Rightarrow \quad = \infty \text{ so rec.}$$

$$\boxed{\mu = \lambda \Rightarrow \sum \prod \frac{\mu's}{\lambda's} \approx \sum n^{\frac{\gamma-\nu}{\lambda}} = \sum \frac{1}{n^{\frac{\nu-\gamma}{\lambda}}}}$$

So
$$\boxed{\nu - \gamma > \lambda \Rightarrow \text{transient}}$$
$$\nu - \gamma \le \lambda \Rightarrow \text{rec.}^{(*)}$$

Pos. v.s. null: similar, but now $\approx \sum n^{\frac{\nu}{\lambda}-\frac{\gamma}{\mu}}\left(\frac{\lambda}{\mu}\right)^n$

so $$\boxed{\mu > \lambda \Rightarrow \sum \prod \frac{\lambda's}{\mu's} < \infty \text{ so Pos. rec.}}$$

$$\mu = \lambda \Rightarrow \sum \prod \frac{\lambda's}{\mu's} \approx \sum \frac{1}{n^{\frac{\gamma-\nu}{\lambda}}}$$

Recall that $\nu - \gamma \le \lambda$ (from (*))

if $\nu - \gamma < -\lambda$ then $\frac{\gamma - \nu}{\lambda} > 1$ and $\sum \prod \frac{\lambda\text{'s}}{\mu\text{'s}} < \infty$

so $\lambda = \mu$ and $\nu - \gamma < -\lambda \Rightarrow$ pos. rec.

lastly: $-\lambda \le \nu - \gamma \le \lambda \Rightarrow$ null rec.

<u>Summary:</u>

$\lambda > \mu \Rightarrow$ transience (pos. chance at survival)

$\lambda < \mu \Rightarrow$ pos. rec. (fast extinction)

$\mu = \lambda < \nu - \gamma \Rightarrow$ transience

$\mu = \lambda < \gamma - \nu \Rightarrow$ pos. rec.

$|\nu - \gamma| \le \lambda = \mu \Rightarrow$ null rec.

$$\overset{\mu n}{\longleftarrow} \bullet \overset{\lambda n}{\longrightarrow} \qquad p(\text{extinction} \mid X_0 = n) = ?$$

$\alpha(n)$!

$$\alpha(0) = 1$$

$$\alpha(n) = \frac{\mu}{\mu+\lambda}\alpha(n-1) + \frac{\lambda}{\mu+\lambda}\alpha(n+1)$$

$$u^n = \frac{\mu}{\mu+\lambda}u^{n-1} + \frac{\lambda}{\mu+\lambda}u^{n+1}$$

$$\lambda u^2 - (\mu+\lambda)u + \mu = 0$$

$u=1$ sol. so $(u-1)(\lambda u - \mu) = 0$

$u = \frac{\mu}{\lambda}$ is the other.

$$\alpha(n) = A + B\left(\frac{\mu}{\lambda}\right)^n$$

[assuming $\mu < \lambda$
o/w $\alpha(n) = 1$]

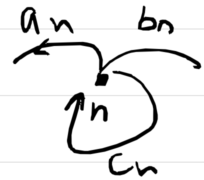
$\alpha(n) \xrightarrow[n\to\infty]{} 0$ means $A = 0$

$\alpha(0) = 1$ means $B = 1$

so $\alpha(n) = \left(\frac{\mu}{\lambda}\right)^n \leftarrow P(\text{extinction} / X_0 = n)$

Recall, we found more generally:

$$\alpha(n) = \frac{\sum_{i=n}^{\infty} \frac{b_1 \cdots b_i}{a_1 \cdots a_i}}{\sum_{i=0}^{\infty} \frac{b_1 \cdots b_i}{a_1 \cdots a_i}}$$



Here, redoing the calculation gives:

$$\alpha(n) = \frac{\sum_{i=n}^{\infty} \frac{\mu_1 \cdots \mu_i}{\lambda_1 \cdots \lambda_i}}{\sum_{i=0}^{\infty} \frac{\mu_1 \cdots \mu_i}{\lambda_1 \cdots \lambda_i}}$$

cancel out in ratios

for $\mu_n = n\mu$ & $\lambda_n = n\lambda$: $\alpha(n) = \dfrac{\sum_{i=n}^{\infty} \left(\frac{\mu}{\lambda}\right)^i}{\sum_{i=0}^{\infty} \left(\frac{\mu}{\lambda}\right)^i}$

$$\alpha(n) = \frac{\left(\frac{\mu}{\lambda}\right)^n \Big/ \left(1-\frac{\mu}{\lambda}\right)}{1 \Big/ \left(1-\frac{\mu}{\lambda}\right)} = \left(\frac{\mu}{\lambda}\right)^n$$

Prob. of survival: $\underline{\underline{1-\left(\frac{\mu}{\lambda}\right)^n}}$

$n=1$ (starting w/ one individual): $\alpha(1) = \frac{\mu}{\lambda}$ (assuming $\mu<\lambda$)

For M/M/1: $\mu_n = \mu$ $\lambda_n = \lambda$

same answer $\left(\frac{\mu}{\lambda}\right)^n$

prob. queue ever empties if started at n ($\lambda > \mu$)