Lec 35

When $\underline{\underline{k\mu > \lambda}}$ (pos. rec.) :

$$\pi(n) = C\,\frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \begin{cases} C\,\dfrac{\lambda^n}{n!\,\mu^n} & \text{if } 0 \le n \le k-1 \\ C\,\underbrace{\dfrac{\lambda^n}{k!\,\mu^k\,k^{n-k}\,\mu^{n-k}}}_{\frac{k^k}{k!}\left(\frac{\lambda}{k\mu}\right)^n} & \text{if } n \ge k \end{cases}$$

$$C\left[\sum_{n=0}^{k-1} \frac{(\lambda/\mu)^n}{n!} + \frac{k^k}{k!}\sum_{n=k}^{\infty}\left(\frac{\lambda}{k\mu}\right)^n\right] = 1$$

$K=1 \quad \pi(n) = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \quad \text{geo}\left(1-\frac{\lambda}{\mu}\right) \quad n = 0, 1, 2, \cdots$

avg. queue length $= \dfrac{\lambda/\mu}{1-\frac{\lambda}{\mu}} = \dfrac{\lambda}{\mu-\lambda}$.

$E_n[T_n] = ? \quad \pi(n) = \dfrac{1/a(n)}{E_n[T_n]}$ ← time spent at n before jumping out
← time to return n once you leave

so $E_n[T_n] = \dfrac{1}{a(n)\,\pi(n)}$

$\pi(0) = C, \; a(0) = \lambda$

$$\boxed{E_0[T_0] = \frac{1}{\lambda C} = \frac{1}{\lambda}\left[\sum_{n=0}^{k-1}\frac{(\lambda/\mu)^n}{n!} + \frac{1}{k!}\cdot\frac{(\lambda/\mu)^k}{1-\frac{\lambda}{k\mu}}\right]}$$

↑ avg. down time b/w empty queues

$k=1: \dfrac{1}{\lambda}\left(1 + \dfrac{\lambda/\mu}{1-\frac{\lambda}{\mu}}\right) = \dfrac{\mu}{\lambda(\mu-\lambda)}$ ($\lambda < \mu$ so that M/M/1 is pos. rec.)

$$E_0[T_0] = E_0[\text{time of first customer arriving}]$$

$$+ E_0[\text{busy period}]$$

$$= \frac{1}{\lambda} + E_0[\text{busy period}]$$

so $E_0[\text{busy period}] = \dfrac{\mu}{\lambda(\mu-\lambda)} - \dfrac{1}{\lambda}$

$$= \frac{1}{\mu-\lambda}$$

can calculate directly using linear difference eqns:

$E[\text{time to get back to } 0 \mid \text{started at }]$

Say $b_x = E[\text{time to get back to } 0 \mid \text{started at } x]$

$$b_0 = 0, \quad x \geq 1: \; b_x = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} b_{x+1} + \frac{\mu}{\lambda+\mu} b_{x-1}$$

prob. jump to $x+1$

avg. time till jumping

Solve and $E_0[\text{busy period}] = b_1$.

$M/M/\infty$: $\lambda_n = \lambda \quad \mu_n = n\mu$

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!\,\mu^n} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n = e^{\lambda/\mu} < \infty \quad \forall \lambda, \mu$$

Always pos rec. (stable queue)

$$\pi(n) = C \cdot \frac{\lambda^n}{n!\,\mu^n}$$

$$= C \cdot \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \qquad C = e^{-\frac{\lambda}{\mu}}$$

inv. meas. is $\mathrm{Poi}\left(\frac{\lambda}{\mu}\right)$

avg. queue length: $\frac{\lambda}{\mu}$.

$$E_0[T_0] = \frac{1}{\lambda \pi(0)} = \frac{1}{\lambda} e^{\frac{\lambda}{\mu}} \qquad E_0[\text{busy period}] = \frac{1}{\lambda}\left(e^{\frac{\lambda}{\mu}} - 1\right)$$

<u>Branching</u> w/o immigration or migration

$\lambda_n = n\lambda \qquad \mu_n = n\mu$

$$\sum_{n=0}^{\infty} \frac{n!\,\mu^n}{n!\,\lambda^n} = \sum_{n=0}^{\infty} \left(\frac{\mu}{\lambda}\right)^n$$

$\mu < \lambda \Rightarrow$ transient

$\mu \geq \lambda \Rightarrow$ recurrent

$$\sum_{n=0}^{\infty} \frac{n!\,\lambda^n}{n!\,\mu^n} = \sum_{n=0} \left(\frac{\lambda}{\mu}\right)^n$$

$\mu > \lambda \Rightarrow$ pos rec.

$\mu = \lambda \Rightarrow$ null rec.