Lec 27

Continuous-time MC: $\{X_t : t \geq 0\}$

still valued in a countable space (finite or infinite) — X_t

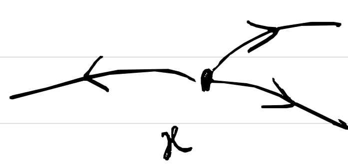
real-valued — t

Markov: $t > s$

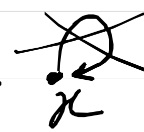
$$P(X_t = y \mid X_r : 0 \leq r \leq s) = P(X_t = y \mid X_s)$$

time homogenous: $P(X_t = y \mid X_s = x) = P(X_{t-s} = y \mid X_0 = x)$

Start w/ a graph:



no loops



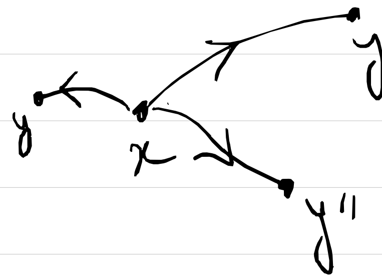
For each

specify $\alpha(x,y) > 0$
(instead of transition probability $p(x,y)$)

attach iid $\exp(\underbrace{\alpha(x,y)}_{\text{rate}})$ random variables.

Think of these as "alarm clocks"
When one rings, the process moves from x to y & the random variable is "discarded"

Now, the story continues, out of y.

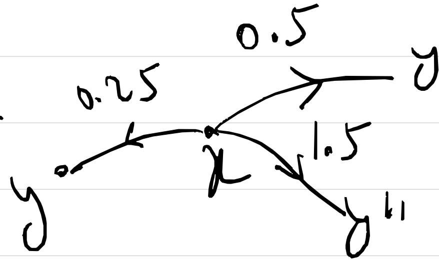


← here, there are three alarm clocks, one for each arrow out of x. Three, possibly different, rates: $\alpha(x,y)$, $\alpha(x,y')$, & $\alpha(x,y'')$.

Whichever one "rings first" signals where the chain moves out of x.

Precisely: three exp random variables. The smallest tells where and when to move.

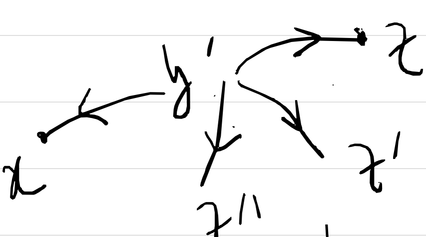
e.g.



rates: 0.25, 0.5, 1.5
random variables:
e.g. 0.758, $\underline{\underline{0.623}}$, 0.924
↑
smallest value

If time now is s, chain goes to y' at time $\underline{\underline{s+0.623}}$

From y':



same story.

Suppose we return to x.
Now, generate three $\underline{\text{new}}$ exp. times.

Q. Is this Markovian?

A. Yes, b/c of the memoriless property of the exp. distribution:

restarting clocks

$$(T_1 - s, T_2 - s, \dots T_k - s \mid \text{all} > 0) \sim (T_1 \dots T_k)$$

iid exp (w/ possibly different rates)

Since these are all indep., enough to show for one:

$(T - s \mid T > s)$ is exp. w/ same rate as T:

Intuition:

exp. can be approximated by geometric.

For geometric: $X - n \mid X > n$ is again a geometric

b/c: $X > n$ means failures

$X - n$ means we restart counting till next success.

This is the same as waiting for the next success!

Now, pf of memoryless property:

$$P(T-s>t \mid T>s) = \frac{P(T>s+t, T>s)}{P(T>s)}$$

$$= \frac{P(T>t+s)}{P(T>s)} = \frac{e^{-\alpha(t+s)}}{e^{-\alpha s}} = e^{-\alpha t} = P(T>t)$$

So $T-s \mid T>s$ has same CDF as T.

So yes, the described process is Markov & time-homogenous.

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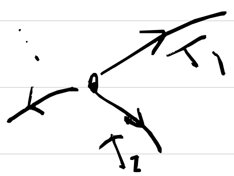
properties of exp. random variable

$T_i \sim \exp(b_i)$ all indep.

$\underbrace{\min(T_1, \ldots, T_n)} \sim \exp(b_1 + \cdots + b_n)$



→ time of jumping out of x is exp(sum of rates out of x)

Intuition n "servers" that serve $b_1, \ldots, b_n$ customers, respectively, per unit time.
min = all working at same time.
rate: $b_1 + \cdots + b_n$ customers per unit time.

proof: $P(\min(T_1, \cdots, T_n) > t) = P(T_1 > t, \cdots, T_n > t)$

$$= P(T_1 > t)\, P(T_2 > t) \cdots P(T_n > t)$$

$$= e^{-b_1 t} e^{-b_2 t} \cdots e^{-b_n t}$$

$$= e^{-(b_1 + \cdots + b_n) t}$$

So $\min(T_1, \cdots, T_n) \sim \exp(b_1 + \cdots + b_n)$.

$T \sim \exp(a)$, $S \sim \exp(b)$ indep.

$$P(T < S) = \iint_{0<t<s} a e^{-at} b e^{-bs} \, dt\, ds$$

$$= a \int_0^\infty e^{-at} \left(b \int_t^\infty e^{-bs} ds \right) dt$$

$$= a \int_0^\infty e^{-at} e^{-bt} dt = \frac{a}{a+b}$$

So $P(\min(T_1, \cdots, T_n) = T_i) = \dfrac{b_i}{b_1 + \cdots + b_n}$

Summary:

$P(\text{jump from } x \text{ to } y)$ [when jump happens]

$$= \frac{\alpha(x,y)}{\sum_{z \in S} \alpha(x,z)}$$

jump out of x: $\exp\left(\sum_{z \in S} \alpha(x,z)\right)$