Lec 26

pos. rec. ?

$$x-1 \xrightarrow{1-\frac{1}{x-1}} x$$

$$\pi(x) = \left(1 - \frac{1}{x-1}\right)\pi(x-1) \qquad x \geq 3$$

$$\pi(2) = \pi(1), \quad \pi(1) = \pi(0)$$

$$\pi(0) = \sum_{x=2}^{\infty} \frac{1}{x}\,\pi(x)$$

$$\pi(x) = \prod_{n=2}^{x-1}\left(1 - \frac{1}{n}\right) \qquad x \geq 3$$

$$= \prod_{n=2}^{x-1} \frac{n-1}{n} = \frac{1}{2}\cdot\frac{2}{3}\cdots\frac{x-2}{x-1}$$

$$= \frac{1}{x-1}$$

$$\sum_{x=3}^{\infty}\pi(x) = \sum_{x=3}^{\infty}\frac{1}{x-2} = \infty \quad \text{so no } \pi!$$

null rec.

what about $0 \to 1 \to 2$, $\quad x \xrightarrow{\frac{1}{\sqrt{x}}} 0$, $\quad x \xrightarrow{1-\frac{1}{\sqrt{x}}} x+1$, $\quad x \geq 2$

again $\sum_{x=2}^{\infty}\frac{1}{\sqrt{x}} = \infty$ so recurrent

$$\pi(x) = \left(1 - \frac{1}{\sqrt{x-1}}\right)\pi(x-1) \qquad x \geq 3$$

$$= \prod_{n=2}^{x-1}\left(1 - \frac{1}{\sqrt{n}}\right)\pi(2) \qquad \pi(2) = \pi(1) = \pi(0)$$

so is $\sum_{x\geq 3}\prod_{n=2}^{x-1}\left(1 - \frac{1}{\sqrt{n}}\right) < \infty$ or ∞ ?

$$\prod_{n=2}^{x-1}\left(1-\frac{1}{\sqrt{n}}\right) = e^{\sum_{n=2}^{x-1} \underbrace{\log\left(1-\frac{1}{\sqrt{n}}\right)}_{\sim -\frac{1}{\sqrt{n}}}}$$

$$\sim -\sum_{n=2}^{x-1}\frac{1}{\sqrt{n}} \sim -\int_2^{x-1}\frac{1}{\sqrt{s}}\,ds$$

$$\approx -\sqrt{x-1}$$

So, $\prod_{n=2}^{x-1}\left(1-\frac{1}{\sqrt{n}}\right) \sim e^{-\sqrt{x-1}}$

$\sum_{x=3}^{\infty}\prod_{n=2}^{x-1}\left(1-\frac{1}{\sqrt{n}}\right)$ is like $\sum_{x=3}^{\infty} e^{-\sqrt{x-1}} < \infty$

$\Big(e^{-\sqrt{x}} \le \frac{1}{x^2}$ for x large

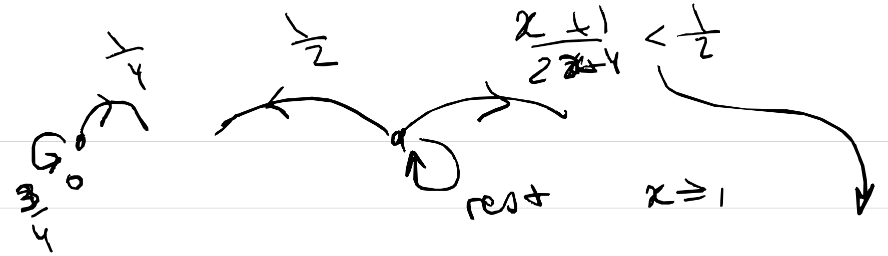
b/c $e^{-y} \le \frac{1}{y^4}$ b/c $y^4 \le e^y$ for y large $\Big)$

So π does exist. Pos. rec.

HW pb 5: $x-1 \xleftarrow{\frac{1}{2}\left(1-\frac{1}{x}\right)} x \xrightarrow{\frac{1}{2}\left(1+\frac{1}{x}\right) > \frac{1}{2}} x+1$

trans.

We'll now see a MC w/ almost the same transition probabilities and yet, pos. rec.!
So it's all very delicate.



birth-death either trans. or null rec.

pos rec. $\pi(x) = \pi(0) \dfrac{a_0 \cdots a_{x-1}}{b_1 \cdots b_x} \quad x \geq 1$

$$= \pi(0) \frac{\prod_{k=0}^{x-1} \frac{k+1}{2k+4}}{\left(\frac{1}{2}\right)^x}$$

$$= \pi(0) \prod_{k=0}^{x-1} \frac{2k+2}{2k+4} = \pi(0) \frac{2}{\cancel{4}} \cdot \frac{\cancel{4}}{\cancel{6}} \frac{\cancel{6}}{\cancel{8}} \cdots \frac{\cancel{2x}}{2x+2}$$

$$= \pi(0) \cdot \frac{1}{x+1}$$

$$\sum_{x=0}^{\infty} \pi(x) = \pi(0) \left(1 + \underbrace{\sum_{x=0}^{\infty} \frac{1}{x+1}}_{\infty}\right)$$ so no $\underline{\pi}$.

Indeed: either transient or null rec.

Transience v.s. Rec : $\displaystyle\sum_{k=1}^{\infty} \frac{b_1 \cdots b_{k-1}}{a_1 \cdots a_{k-1}} = \sum_{k=1} \frac{\left(\frac{1}{2}\right)^{k-1}}{\prod_{i=1}^{k-1} \frac{i+1}{2i+4}}$

$$= \sum_{k=1}^{\infty} \underbrace{\prod_{i=1}^{k-1} \frac{2i+4}{2i+2}}_{} \rightarrow \frac{\cancel{6}}{4} \cdot \frac{\cancel{8}}{\cancel{6}} \cdots \frac{2k+2}{\cancel{2k}} = \frac{k+1}{2}$$

$\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{2} = \infty$ so Rec.

Since not pos. rec. $\underline{\text{Null Rec.}}$