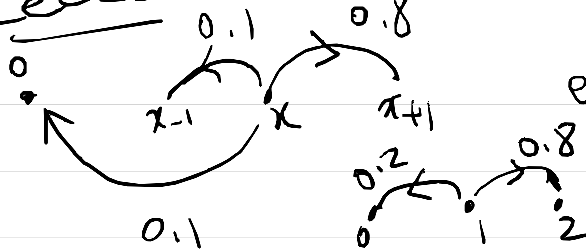
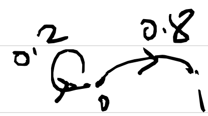
Lec25




expect: pos. rec.




$$(*)\quad \pi(x) = 0.8\,\pi(x-1) + 0.1\,\pi(x+1) \qquad x \geq 1$$

$$\pi(0) = 0.2\,\pi(0) + 0.2\,\pi(1) + 0.1 \underbrace{\sum_{x=2}^{\infty} \pi(x)}_{1-\pi(0)-\pi(1)}$$

$$\pi(0) = 0.2\,\pi(0) + 0.2\,\pi(1) + 0.1 - 0.1\,\pi(0) - 0.1\,\pi(1)$$

$$0.9\,\pi(0) = 0.1 + 0.1\,\pi(1)$$

$$\boxed{\pi(1) = 9\,\pi(0) - 1}$$

(*) is a linear diff. eqn. so:

$$0.1u^2 - u + 0.8 = 0$$

$$\Delta = \sqrt{1-0.32} = \sqrt{0.68} \approx \sqrt{\frac{64}{100}} \approx \frac{8}{10} = 0.8$$

$$u_\pm = \frac{1 \pm \sqrt{0.68}}{0.2} \approx \frac{1 \pm 0.8}{0.2} = \begin{cases} \leq 1 & \leftarrow u_- \\ \gtrsim 9 & \leftarrow u_+ \end{cases}$$

$$\pi(x) = A u_-^x + B \underbrace{u_+^x}_{\to\infty \text{ as } x\to\infty} \qquad \text{so } B = 0$$

$$\pi(x) = A u_-^x \qquad x \geq 1$$

$$1 = \pi(0) + \sum_{x=1}^{\infty} A u_-^x = \pi(0) + A\frac{u_-}{1-u_-}$$

$\pi(1) = A u_-$

so $$1 = \pi(0) + \frac{\pi(1)}{1-u_-} = \pi(0) + \frac{9\pi(0)-1}{1-u_-}$$

$$(10-u_-)\pi(0) = 2-u_-$$

$$\pi(0) = \frac{2-u_-}{10-u_-} \sim \frac{1}{9} > 0$$

$$\pi(1) = 9\pi(0) - 1 \geq 0 ?$$

$$= \frac{18-9u_-}{10-u_-} - 1 = \frac{8(1-u_-)}{10-u_-} \qquad \underline{\underline{u_- \lneqq 1}}$$

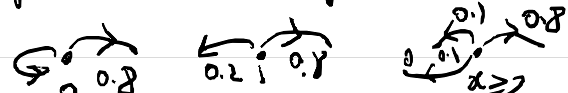
so $\geq 0$

$x \geq 1 \quad \pi(x) = \pi(1) u_-^{x-1}$ ✓

so pos. rec.

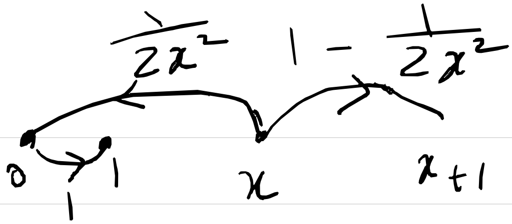
Faster: $P_0(T_0 \geq n) \leq (0.9)^n$

explanation: $n$ indep. steps that don't take us to 0 each time, prob. to not go back to 0 is $\geq 0.9$ :




$0.8 \leq 0.9 \quad 0.8 \leq 0.9 \quad 0.9 \leq 0.9$

so $$E[T_0] = \sum_{n=0}^{\infty} P(T_0 \geq n) \leq \sum_{n=0}^{\infty} 0.9^n < \infty$$




transient?

$$\alpha(x) = \frac{1}{2x^2}\,\alpha(0) + \left(1 - \frac{1}{2x^2}\right)\alpha(x+1) \quad ?$$

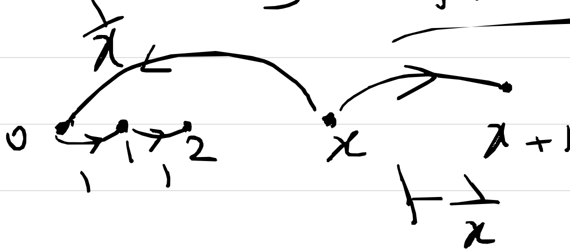
$$P_0(\text{never return to } 0) = \prod_{x=1}^{\infty}\left(1 - \frac{1}{2x^2}\right)$$

$$= e^{\sum_{x=1}^{\infty} \underbrace{\log\left(1 - \frac{1}{2x^2}\right)}_{\sim -\frac{1}{2x^2}}}$$

$$\sum_{x=1}^{\infty} \frac{1}{2x^2} < \infty \text{ so } e^{-\text{some neg. \#}}$$

$$P_0(\text{never return to } 0) > 0$$

So <u>transience</u>




same reasoning: $P_0(\text{never return})$

$$= \prod_{x=2}^{\infty}\left(1 - \frac{1}{x}\right) = e^{\sum_{x=2}^{\infty} \underbrace{\log\left(1 - \frac{1}{x}\right)}_{\sim -\frac{1}{x}}}$$

now $\sum_{x=2}^{\infty} \frac{1}{x} = \infty$

So $P_0(\text{never return}) = e^{-\infty} = 0$

Recurrence!