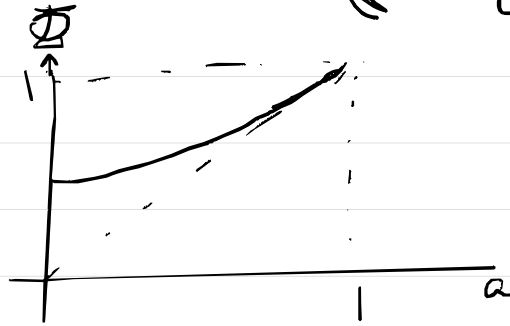
Lec 24

Back to $\Phi(s)=s$ (doing the calculus)

Case 1 $\mu=1$:



sol. of $\Phi(a)=a$ in $[0,1]$ is unique: $a=1$.

Pf. $\Phi''>0$ so Φ' is strictly increasing

$\Phi'(1)=\mu=1$ so $\Phi'(s)<1$ $\forall s\in[0,1)$

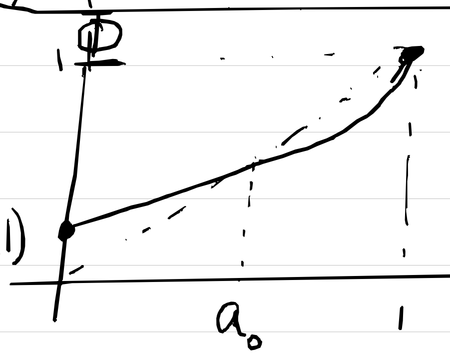
so $(\Phi(s)-s)'<0$ and $\Phi(s)-s$ is strictly $\nearrow$

$\Phi(1)=1$ so $\Phi(s)-s<0$ $\forall s\in[0,1)$

i.e. 1 is the only sol. to $\Phi(s)=s$

So $\boxed{\text{when } \mu=1,\ P(\text{extinction})=1}$

Case 2 $\mu>1$:



two solutions: $a=1$ and $a=a_0\in(0,1)$

Pf. $\phi''>0$ so ϕ' is strictly $\nearrow$

$\phi'(1)=\mu>1$ while $\phi'(0)=P(Z=1)$

$$\phi'(s)=\sum_{k=1}^{\infty} k s^{k-1} p(k)$$

$p(1) = P(Z=1) < 1$ (otherwise $X_n = X_0 \ \forall n \geq 1$!)

So $\phi'(s) - 1 < 0$ at 0 and > 0 at 1

and strictly increasing

So:

s	0		s_0		1
$\phi'(s)-1$		$-$	0	$+$	
$\phi(s)-s$	$+$	$\searrow$		$\nearrow$	0

$\phi(0) - 0 = p(0) > 0$ $\qquad \phi(1) - 1 = 0$ (and continuous)

So $\exists a_0 \in (0, s_0)$ s.t. $\phi(a_0) - a_0 = 0$

So <u>two</u> solutions: a_0 & 1.

(and we proved earlier that a_0 is the one we want)

Example:

$P_0=\frac{1}{2}$ $P_1=\frac{1}{4}$ $P_2=\frac{1}{4}$ $\mu=\frac{3}{4}<1$ $a=1$

$P_0=\frac{1}{4}$ $P_1=\frac{1}{2}$ $P_2=\frac{1}{4}$ $\mu=1$ $a=1$

$P_0=\frac{1}{4}$ $P_1=\frac{1}{4}$ $P_2=\frac{1}{2}$ $\mu=\frac{5}{4}>1$ $\underline{\underline{a<1}}$

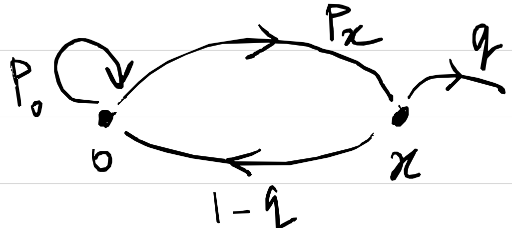
$\hookrightarrow \phi(a)=\frac{1}{4}+\frac{a}{4}+\frac{a^2}{2}$

$\phi(a)=a$: $2a^2-3a+1=0$ $(a-1)(2a-1)=0$

$a=1$ & $\underline{\underline{a=\frac{1}{2}}}$

P(extinction) = $\underline{1/2}$

P(survival) = $1-\frac{1}{2}=\underline{\frac{1}{2}}$



expect: pos. rec.

First: rec. vs. trans.

P(never get back to 0) $=\sum_{x\geq 1} P_x \cdot \underbrace{q^{\infty}}_{0} = 0$

So recurrent.

For pos. rec.: find π.

$$\pi(x) = p_x \pi(0) + q\,\pi(x-1) \qquad x \geq 1$$

How to solve ?! Not a linear difference.

return to 0

Alternative: $E_0[T_0] < \infty$?

leave 0   forward $n-2$ steps

$P(T_0 = 1) = p_0$

$n \geq 2$: $P(T_0 = n) = (1-p_0) \cdot q^{n-2} (1-q)$ back to 0

$$E[T_0] = p_0 + \sum_{n \geq 2} (1-p_0) q^{n-2} (1-q)\, n$$

$$= p_0 + (1-p_0)(1-q) \underbrace{\sum_{n=2}^{\infty} n q^{n-2}}_{<\infty}$$

$$= \sum_{m=1}^{\infty} (m+1) q^{m-1} (1-q)$$

$$= \frac{1}{1-q} + 1 \leftarrow \sum_{m=1}^{\infty} q^{m-1}(1-q)$$

$$= \frac{2-q}{1-q}$$

↑ mean of geometric w/ values 1, 2, ...

$$= p_0 + \frac{(1-p_0)(2-q)}{1-q}$$

$$= \frac{2-p_0-q}{1-q}$$

pos. rec.