Lec23

$$\phi_Z(s) = P(Z=0) + \sum_{k=1}^{\infty} s^k P(Z=k)$$

$\leq \sum_{k=1}^{\infty} s^k = \frac{s}{1-s} \xrightarrow[s\to 0]{} 0$

so $\Phi_Z(0) = P(Z=0)$

$$\Phi_Z(1) = E[1^Z] = 1$$

$\Phi_Z \nearrow$ strictly unless $P(Z=0)=1$

silly: $X_n = 0 \ \forall n \geq 1$

$$\phi_Z'(s) = \sum_{k=1}^{\infty} k s^{k-1} P(Z=k)$$

so $\phi_Z'(1) = \sum_{k=1}^{\infty} k P(Z=k) = E[Z] = \mu$

$$\phi_Z''(s) = \sum_{k=2}^{\infty} k(k-1) s^{k-2} P(Z=k)$$

> 0 unless $P(Z \geq 2) = 0$ so $P(Z \in \{0,1\}) = 1$

but then either $P(Z=1)=1$ & $X_n = 1 \ \forall n$

silly

or $P(Z=1) < 1$ and so

$\mu = E[Z] < 1$ (weighted average of 0 & 1)

not case being considered

Summary:

$\Phi_Z(0) = p(0)$, $\Phi_Z(1) = 1$

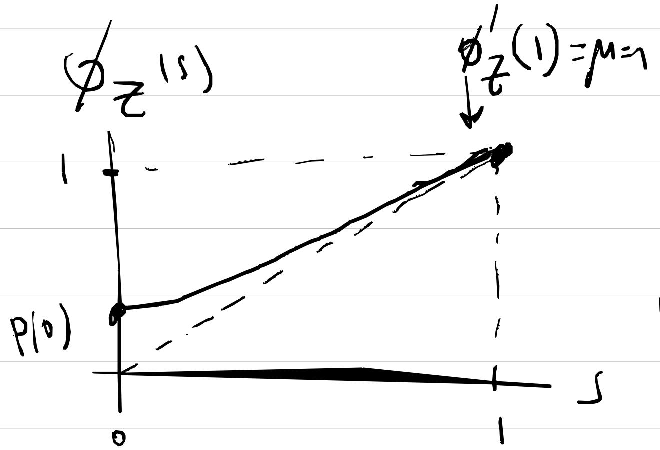
$\Phi_Z'(1) = \mu$, Φ_Z strictly $\nearrow$ ← assumption $p(0) < 1$

$\Phi_Z'' > 0$ ← assumption $p(0) + p(1) < 1$

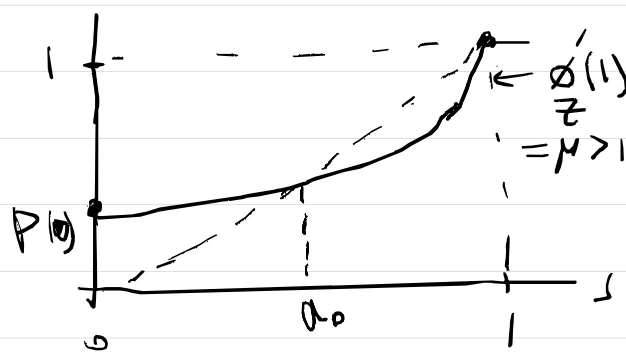
so Φ_Z strictly cvx

Will also assume $p(0) > 0$ ← $P(Z=0) = 0 \Rightarrow X_{n+1} \geq X_n \ \forall n$ so no extinction

Φ_Z is <u>continuous</u>



$\mu = 1$



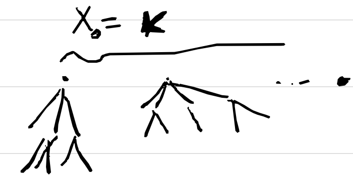
$\mu > 1$

Let $a = P(\exists n : X_n = 0 \mid X_0 = 1)$
← prob. of extinction

Q: $a = 1$ or < 1?

$$P(\text{extinction} \mid X_0 = k) = a^k$$

(each tree is independent:



and each gets extinct w/ prob. a

for extinction, we need all trees to end)

$$a = P(\text{extinction} \mid X_0 = 1) = \sum_{k=0}^{\infty} p(k) \underbrace{P(\text{ext.} \mid X_1 = k)}_{\substack{= P(\text{ext.} \mid X_0 = k) \\ = a^k}}$$

So a solves:

$$\boxed{\Phi(s) = s, \ s \in [0,1]} \qquad \underbrace{\sum_{k=0}^{\infty} p(k) a^k}= \Phi(a)$$

So when $\mu = 1$, the only possibility is $a = 1$ (the only solution in $[0,1]$ of $\Phi(s) = s$)

But when $\mu > 1$, there are two options: $a = a_0 \in (0,1)$ or $a = 1$

Which one is it?

will prove in a bit

for now: from picture

Say $\mu > 1$:

Is $P(\text{extinction}) = 1$ or a_0 ??

both solve $\Phi(s) = s$

$$E\left[a_0^{X_{n+1}} \mid X_n = k\right] = E\left[a_0^{Z_1^{(n)} + \dots + Z_k^{(n)}}\right]$$

$$= E\left[a_0^{Z_1^{(n)}} \cdots a_0^{Z_k^{(n)}}\right]$$

indep.

$$= E\left[a_0^{Z_1^{(n)}}\right] \cdots E\left[a_0^{Z_k^{(n)}}\right] = \Phi(a_0)^k = a_0^k$$

b/c $\Phi(a_0) = a_0$

So by law of total prob.:

$$E\left[a_0^{X_{n+1}}\right] = \sum_{k=0}^{\infty} P(X_n = k)\, E\left[a_0^{X_{n+1}} \mid X_n = k\right]$$

$$= \sum_{k=0}^{\infty} P(X_n = k)\, a_0^k = E\left[a_0^{X_n}\right]!$$

So $E\left[a_0^{X_n}\right] = E\left[a_0^{X_{n-1}}\right] = \dots = E\left[a_0^{X_0}\right]$

$X_0 = 1 \rightarrow\ = a_0$

$$P(X_n = 0) \le \sum_{k \ge 0}^{\infty} a_0^k \, P(X_n = k)$$

b/c
$k=0$ gives
$P(X_n=0)$
and all other
k give ≥ 0

$$= E\left[a_0^{X_n}\right] = a_0$$

$n \to \infty$: $P(\text{extinction}) \le a_0 < 1$

So not 1.

Only other option now: $P(\text{extinction}) = a_0$

<u>Summary:</u>

1) $\mu < 1$: $P(\text{extinction}) = 1$

Includes: $P(Z=0)=1$ & $P(Z=0)+P(Z=1)=1$
w/ $P(Z(0)=0)>0$

2) $\mu \ge 1$:

Silly case: $P(Z=0)=0 \to X_{n+1} \ge X_n$ $P(\text{no ext.}) = 1$

So now: $0 < P(Z=0) < 1$ & $P(Z=0)+P(Z=1) < 1$

Then: $\mu = 1 \to P(\text{extinction}) = 1$

$\mu > 1 \Rightarrow P(\text{extinction}) = a \in (0,1)$

a solves $\Phi(a) = a$