Lec 22

$$\mu = \sum_{i=0}^{\infty} i\,p(i)$$ mean # of descendents per individual

Intuition:

$\mu > 1 \Rightarrow$ exponential growth (transience)

$\mu < 1 \Rightarrow$ fast extinction (pos rec.)

$\mu = 1 \Rightarrow$? extinction or not ?

null recurrence ? ← extinction but $E[N] = \infty$

(If N = # generations to extinction)

Won't use α & π!

Note: $\mu = E[Z_i^n]$ $\forall i$ $\forall n$

So given $X_n = x$:

$$E[X_{n+1} \mid X_n = x] = E\left[\sum_{i=1}^{x} Z_i^n\right] = \mu x$$

So $E[X_{n+1}] = \sum E[X_{n+1} \mid X_n = x]\, P(X_n = x)$

$$= \mu \sum_x x\, P(X_n = x) = \mu E[X_n]$$

$$E[X_{n+1}] = \mu E[X_n]$$

gives $E[X_n] = \mu E[X_{n-1}] = \mu^2 E[X_{n-2}]$

$= \cdots = \mu^n E[X_0]$

So $\mu > 1 \Rightarrow E[X_n]$ grows exp. fast

$\mu < 1 \Rightarrow E[X_n]$ decays exp. fast

$\mu = 1 \Rightarrow E[X_n]$ remains constant

Enough to study $X_0 = 1$

(b/c each individual behaves independently and identically!)

Then $P(X_n \geq 1) = \sum_{x \geq 1} P(X_n = x) \leq \underbrace{\sum_{x \geq 1} x P(X_n = x)}_{E[X_n] = \mu^n}$

so $P(X_n \geq 1) \leq \mu^n$

Say $\mu < 1$. $\underbrace{\sum_{n=0}^{\infty} P(X_n \geq 1)}_{E[\sum_{n=0}^{\infty} 1\{X_n \geq 1\}]} \leq \sum_{n=0}^{\infty} \mu^n = \frac{1}{1-\mu} < \infty$

So if $\mu < 1$ then $\sum_{n=0}^{\infty} 1\{X_n \geq 1\} < \infty$

so eventually (for n large enough)

$X_n = 0$ <u>extinction</u>.

In fact $\sum_{n=0}^{\infty} 1\{X_n \geq 1\} = N$

\# generations till extinction

and we've shown

$$E[N] \leq \frac{1}{1-\mu} < \infty \quad \text{(positive recurrence)}$$

Now we focus on $\mu \geq 1$.

<u>Tool</u>. probability generation fn

Z has values $0, 1, 2, \ldots$

$$\Phi_Z(s) = E[s^Z] = \sum_{k=0}^{\infty} s^k \, p(Z=k) \qquad 0 \leq s \leq 1$$

e.g. $Z \sim \mathrm{Poi}(\lambda)$ $\quad \Phi_Z(s) = \sum_{k=0}^{\infty} s^k \frac{\lambda^k}{k!} e^{-\lambda} = e^{s\lambda - \lambda} = e^{\lambda(s-1)}$