Lec 21

Note:

return time to x

$$E_0[T_0] = \frac{1}{\pi(0)}$$

$$= \sum_{i=0}^{\infty} \frac{a_0 \cdots a_{i-1}}{b_1 \cdots b_i}$$

(again; pos. rec. $\Leftrightarrow$ $E_0[T_0] < \infty \Leftrightarrow$ $< \infty$)

Queues: $a_x = p(1-q)$ $b_x = q(1-p)$ (for $x \geq 1$)

Transience v.s. Recurrence:

$$\sum_{i=1}^{\infty} \frac{b_1 \cdots b_{i-1}}{a_1 \cdots a_{i-1}} = 1 + \sum_{i=2}^{\infty} \frac{b_1 \cdots b_{i-1}}{a_1 \cdots a_{i-1}}$$

$$= 1 + \sum_{i=2}^{\infty} \left(\frac{q(1-p)}{p(1-q)}\right)^{i-1}$$

$$= \begin{cases} \dfrac{1}{1 - \dfrac{q(1-p)}{p(1-q)}} & \text{if } \dfrac{q(1-p)}{p(1-q)} < 1 \\ \infty & \text{o/w} \end{cases}$$

transience

So $q(1-p) < p(1-q)$: $(q < p)$

$q \geq p$: recurrence.

Positive v.s. null recurrence:

$$\sum_{i=0}^{\infty} \frac{a_0 \cdots a_{i-1}}{b_1 \cdots b_i} = 1 + \sum_{i=1}^{\infty} \frac{a_0 \cdots a_{i-1}}{b_1 \cdots b_i}$$

$$= 1 + \sum_{i=1}^{\infty} \frac{p \cdot (p(1-q))^{i-1}}{(q(1-p))^i}$$

$a_0 = p$

$a_x = p(1-q)$ if $x \geq 1$

$$= 1 + \frac{p}{q(1-p)} \underbrace{\sum_{i=1}^{\infty} \left(\frac{p(1-q)}{q(1-p)}\right)^{i-1}}$$

if $p<q$

$$= 1 + \frac{p}{q(1-p)} \cdot \frac{1}{1 - \frac{p(1-q)}{q(1-p)}}$$

$$= 1 + \frac{p}{q(1-p) - p(1-q)} = 1 + \frac{p}{q-p}$$

$$= \frac{q}{q-p} \quad (*)$$

$$\begin{cases} \dfrac{1}{1 - \dfrac{p(1-q)}{q(1-p)}} & \text{if } \dfrac{p(1-q)}{q(1-p)} < 1 \\ \infty & \text{o/w} \end{cases}$$

So Pos. Rec. : $p(1-q) < q(1-p)$

i.e. $\boxed{p<q}$

$p=q$ not pos. rec. & not transient

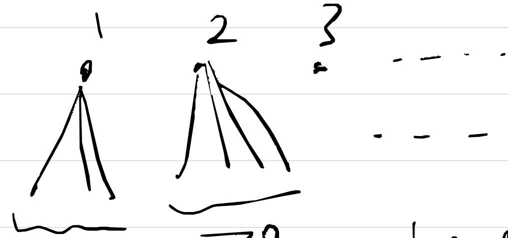
So null recurrent

$p<q \implies$ pos. rec. What's $E_0[T_0]$?

$$E_0[T_0] = \frac{1}{\pi(0)} = \sum_{i=0}^{\infty} \frac{a_0 \cdots a_{i-1}}{b_1 \cdots b_i} \overset{\text{from } (*)}{=} \frac{q}{q-p}$$

Note: $\to \infty$ as $q \to p$!

Branching process



X_0 ← population at generation 0

Z_2^0 ← descendents of individual # 2 gen # 0

Z_1^0 ← descendents of individual #1 in gen #0

$$X_1 = \sum_{i=1}^{X_0} Z_i^0$$ ← population at generation 1

$\vdots$

X_n ← pop. gen. n

$$X_{n+1} = \sum_{i=1}^{X_n} Z_i^n$$ ← iid ~ mass fn p

same for all

↑ # descendents of individual i @ gen n

Markov b/c to generate X_{n+1}

we need to only know X_n

Once we know X_n, we generate

$Z_1^n \cdots Z_{X_n}^n$ iid w/ mass fn p

assumption of model } same for all individuals & generations