Lec 20

So: $a_x(\alpha(x+1)-\alpha(x)) - b_x(\alpha(x)-\alpha(x-1)) = 0$

$x \geq 1$

$$\alpha(x+1)-\alpha(x) = \frac{b_x}{a_x}(\alpha(x)-\alpha(x-1))$$

$$\alpha(x)-\alpha(x-1) = \frac{b_{x-1}}{a_{x-1}}(\alpha(x-1)-\alpha(x-2))$$

$$= \frac{b_{x-1}}{a_{x-1}} \cdot \frac{b_{x-2}}{a_{x-2}}(\alpha(x-2)-\alpha(x-3))$$

$$= \cdots = \frac{b_{x-1}\cdots b_1}{a_{x-1}\cdots a_1}\underbrace{(\alpha(1)-\alpha(0))}_{\text{call it } C}$$

So $\alpha(x)-\alpha(x-1) = C\dfrac{b_1 \cdots b_{x-1}}{a_1 \cdots a_{x-1}}$ $\quad x \geq 2$

$\alpha(1)-\alpha(0) = C$ [so when $x=1$ interpret $\dfrac{b_1 \cdots b_{x-1}}{a_1 \cdots a_{x-1}} = 1$]

$$\alpha(x) = \alpha(0) + C\sum_{i=1}^{x}\frac{b_1\cdots b_{i-1}}{a_1\cdots a_{i-1}}$$

[when $i=1$ $\dfrac{b_1 \cdots b_{i-1}}{a_1 \cdots a_{i-1}} = 1$]

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If $\displaystyle\sum_{i=1}^{\infty}\frac{b_1\cdots b_{i-1}}{a_1\cdots a_{i-1}} = \infty$ then $C=0$ b/c α is bounded

and so $\alpha(x)=1$ for all x

so $\alpha(x) \not\to 0$ as $x \to \infty$

then recurrent

Recall: this is just 1 for i=1

If $\sum_{i=1}^{\infty} \frac{b_1 \cdots b_{i-1}}{a_1 \cdots a_{i-1}} < \infty$ then $1 + C \underbrace{\sum_{i=1}^{\infty} \frac{b_1 \cdots b_{i-1}}{a_1 \cdots a_{i-1}}}_{\lim_{x\to\infty} \alpha(x)} = 0$

so $C = - \dfrac{1}{\sum_{i=1}^{\infty} \frac{b_1 \cdots b_{i-1}}{a_1 \cdots a_{i-1}}}$

and $\alpha(x) = 1 - \dfrac{\sum_{i=1}^{x} \frac{b_1 \cdots b_{i-1}}{a_1 \cdots a_{i-1}}}{\sum_{i=1}^{\infty} \frac{b_1 \cdots b_{i-1}}{a_1 \cdots a_{i-1}}}$

$= \dfrac{\sum_{i=x+1}^{\infty} \frac{b_1 \cdots b_{i-1}}{a_1 \cdots a_{i-1}}}{\sum_{i=1}^{\infty} \frac{b_1 \cdots b_{i-1}}{a_1 \cdots a_{i-1}}}$

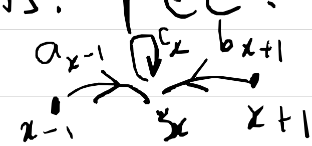
≥ 0, $\to 0$ as $x \to \infty$

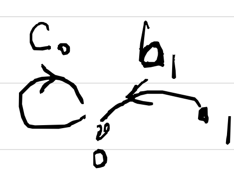
$\alpha(0) = 1$ ✓

so **transient**

Rec. vs. pos. rec.?

Need π.





$\pi(x) = a_{x-1}\pi(x-1) + b_x \pi(x+1) + c_x \pi(x)$

$x \geq 1$

$\pi(0) = b_1 \pi(1) + \underbrace{c_0}_{1-a_0} \pi(0) \Rightarrow \boxed{\pi(1) = \frac{a_0}{b_1}\pi(0)}$

$(a_x + b_x + \cancel{c_x})\pi(x) = a_{x-1}\pi(x-1) + b_{x+1}\pi(x+1) + \cancel{c_x \pi(x)}$

So $b_{x+1}\pi(x+1) - a_x\pi(x) = b_x\pi(x) - a_{x-1}\pi(x-1)$

This gives $b_x\pi(x) - a_{x-1}\pi(x-1)$ same for all $x \geq 1$

so $= C$

$b_1 \underbrace{\pi(1)}_{\frac{a_0}{b_1}\pi(0)} - a_0\pi(0) = C$ so $C = 0$

$\underbrace{\qquad}_{=0}$

Now $b_x\pi(x) - a_{x-1}\pi(x-1) = 0 \quad \forall x \geq 1$

So $\pi(x) = \dfrac{a_{x-1}}{b_x}\pi(x-1)$

$= \cdots = \dfrac{a_{x-1}a_{x-2}\cdots a_0}{b_x b_{x-1}\cdots b_1}\pi(0)$

We want $\sum_{x=0}^{\infty}\pi(x) = 1$

Recall notation: this is 1 when $i=0$ (empty product)

So need: $\sum_{i=0}^{\infty}\dfrac{a_0\cdots a_{i-1}}{b_1\cdots b_i} < \infty$

If $\sum_{i=0}^{\infty} \frac{a_0 \cdots a_{i-1}}{b_1 \cdots b_i} < \infty$ then

$$\pi(x) = \frac{a_0 \cdots a_{x-1}}{b_1 \cdots b_x} \pi(0)$$

$$1 = \sum_{x=0}^{\infty} \frac{a_0 \cdots a_{x-1}}{b_1 \cdots b_x} \pi(0) \Rightarrow \pi(0) = \frac{1}{\sum_{x=0}^{\infty} \frac{a_0 \cdots a_{x-1}}{b_1 \cdots b_x}}$$

so $$\pi(x) = \frac{\frac{a_0 \cdots a_{x-1}}{b_1 \cdots b_x}}{\sum_{i=0}^{\infty} \frac{a_0 \cdots a_{i-1}}{b_1 \cdots b_i}}$$

Positive recurrence.

If $\sum_{i=0}^{\infty} \frac{a_0 \cdots a_{i-1}}{b_1 \cdots b_i} = \infty$

then $1 = \underbrace{\sum_{x=0}^{\infty} \frac{a_0 \cdots a_{x-1}}{b_1 \cdots b_x}}_{\infty} \pi(0)$ so $\pi(0) = 0$ and $\pi(x) = 0$ $\forall x \geq 0$!

So no π that sums to 1.

Not positive recurrent

(either transient or null rec.)