Lec18

$\sum P_n(0,0)$ is not always computable

There are other criteria.

Fix a z (to choose so that computations are simpler)

Define:

$$\alpha(x) = P(\text{ever visit } z \mid X_0 = x)$$

$\alpha(x) = 1 \quad \forall x$ (if recurrent)

If transient:

1) $0 \leq \alpha(x) \leq 1 \quad \forall x$

2) $\alpha(z) = 1$

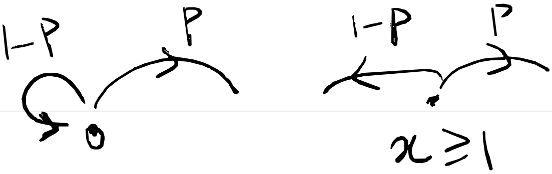
3) $\alpha(x) \to 0$ as x gets "far" from z (so $\inf_x \alpha(x) = 0$)

4) $x \neq z$: $\alpha(x) = \sum_{y \in S} \alpha(y) P(x,y)$

Transient $\Rightarrow$ such α exists also $\Leftarrow$! (not obvious:)

Recurrent $\Leftarrow$ no such α also $\Rightarrow$ (won't prove)

Application



Intuition: $p > \frac{1}{2}$: $\longrightarrow$ so transient

$p < \frac{1}{2}$: $\longleftarrow$ so recurrent

$p = \frac{1}{2}$: SSRW so recurrent

Pf. $z = 0$

$\alpha(0) = 1$

$x \geq 1$: $\alpha(x) = p\,\alpha(x+1) + (1-p)\,\alpha(x-1)$

Other condition: $\alpha(x) \underset{x \to \infty}{\longrightarrow} 0$

Solved before: $\alpha(x) = A + B\left(\frac{1-p}{p}\right)^x$

$\alpha(0) = 1$: $1 = A + B$

$\alpha(x) \underset{x \to \infty}{\longrightarrow} 0$:

$p > \frac{1}{2} \Rightarrow \frac{1-p}{p} < 1 \Rightarrow \left(\frac{1-p}{p}\right)^x \to 0$

So $0 = A$ and $1 = B$

$\alpha(x) = \left(\frac{1-p}{p}\right)^x$ transient ✓

$p > \frac{1}{2} \Rightarrow \frac{1-p}{p} > 1 \Rightarrow \left(\frac{1-p}{p}\right)^x \to \infty$

so $B = 0$ and $A = 1$

$\alpha(x) = 1 \quad \forall x$ does not $\underset{x \to \infty}{\longrightarrow} 0$

so recurrent

$p = \frac{1}{2}$

$\alpha(x) = A + Bx$

$\alpha(0) = 1 \Rightarrow A = 1$

$\alpha(x) \underset{x \to \infty}{\longrightarrow} 0 \Rightarrow B = 0$

So $\alpha(x) = 1 \quad \forall x$

again: recurrent

T_x = time to return to x

$$E[T_x \mid X_0 = x] = ? \begin{cases} \infty & \text{if trans.} \\ ? & \text{if rec.} \end{cases}$$

$= \infty$ null recurrent

$< \infty$ positive recurrent

recurrent but "barely" (takes very long to return)

Finite MC: $E[T_x \mid X_0 = x] = \dfrac{1}{\Phi_\infty(x)} < \infty$

so always pos. rec.

Infinite MC: null rec. or trans. $\Rightarrow$ no inv. meas.!

pos. rec $\Rightarrow$ unique inv. meas.

Will use π instead of Φ_∞