Lec. 16

Stirling: $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$ will show $c\sqrt{n}\, n^n e^{-n}$

Pf $f(n) = \dfrac{n!}{\sqrt{n}\, n^n e^{-n}}$ $\qquad f(1) = e$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + o(|x|^3)$$

$$\log f(n) - \log f(n-1) = \log\left(\frac{n!}{\sqrt{n}\, n^n e^{-n}} \cdot \frac{\sqrt{n-1}\,(n-1)^{n-1} e^{-n+1}}{(n-1)!}\right)$$

$$= \log\left(e \cdot \cancel{n} \sqrt{1-\frac{1}{n}}\left(1-\frac{1}{n}\right)^{n-1} \cdot \frac{1}{\cancel{n}}\right)$$

$$= 1 + \left(n-\frac{1}{2}\right)\underbrace{\log\left(1-\frac{1}{n}\right)}_{-\frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} + o\left(\frac{1}{n^3}\right)}$$

$$= \cancel{1} - \cancel{1} + \cancel{\frac{1}{2n}} - \cancel{\frac{1}{2n}} + \frac{1}{4n^2} - \frac{1}{3n^2} + o\left(\frac{1}{n^2}\right)$$

$$= -\frac{1}{12n^2} + o\left(\frac{1}{n^2}\right)$$

$$\log f(n) = -\frac{1}{12}\sum_{k=2}^{n} \frac{1}{k^2} + \underbrace{\log f(1)}_{\log e = 1} + \underbrace{o\left(\sum_{k=2}^{n} \frac{1}{k^3}\right)}_{\le o(1) \text{ so } \to 0 \text{ as } n\to\infty}$$

$$\xrightarrow[n\to\infty]{} C\left(= 1 - \frac{1}{12}\left(\frac{\pi^2}{6} - 1\right)\right)$$

So $f(n) \sim e^c$ and $n! \sim C'\sqrt{n}\, n^n e^{-n}$

Why $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$? (actually $= \frac{\pi^2}{6}$) $\qquad 1 + \sum_{k=2}^{\infty} \frac{1}{k^2} \le 1 + \sum_{k=2}^{\infty} \underbrace{\frac{1}{k(k-1)}}_{\frac{1}{k-1} - \frac{1}{k}} = 1+1$

$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} \cdots \longrightarrow$