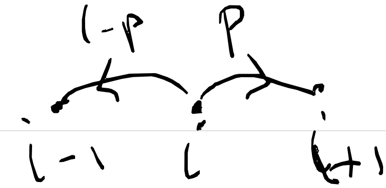
Lec 15

**Example RW**

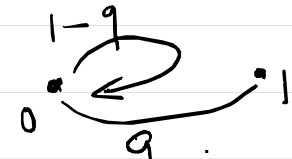



Could be on all $\mathbb{Z}$

or on $0, 1, \ldots$

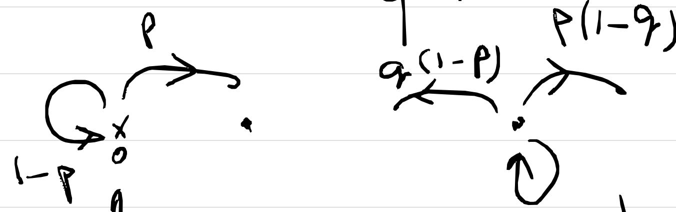
with e.g. $0$ ⟲ prob. 1 (absorbed)

or




(reflected)

**Queue:**




$q$ — empty queue

rest: $pq + (1-p)(1-q)$

$= 1 - q(1-p) - p(1-q)$

[ new customer: prob. $p$

customer leaves: prob. $q$ ]

each time (discrete time)

**Q** Queue grows indefinitely?

Or empties eventually?

How long till it empties?

different from finite state space even though irreducible!

**Recurrent**: $P(\text{return to } x \mid X_0 = x) = 1$

**Transient** $P(\text{return to } x \mid X_0 = x) < 1$

**Th** MC irreducible (so enough to look at rec. vs. tran. at one site)

Transient iff $\sum_{n=1}^{\infty} P_n(x,x) < \infty$

$$\sum_{n=1}^{\infty} P(X_n = x \mid X_0 = x) = \sum_{n=1}^{\infty} E_x[1\{X_n = x\}] = E_x\Big[\underbrace{\sum_{n=1}^{\infty} 1\{X_n = x\}}_{\#\text{returns to } x}\Big]$$

Pf Transient

$\Updownarrow$

$\alpha = P(\text{never return to } x \mid X_0 = x) > 0$

#returns to $x$ is geo($\alpha$)

(return $\leftrightarrow$ failure)

so transient $\Leftrightarrow$ $E_x[\#\text{returns to } x] = \frac{1}{\alpha} < \infty$

$\square$

**Application** SSRW $\frac{1}{2}$ $\frac{1}{2}$

Choose $x = 0$, $P_n(0,0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ ? & \text{if } n \text{ even} \end{cases}$

To return to 0 in $2n$ steps need $n$ left & $n$ right

so $P_{2n}(0,0) = \binom{2n}{n} \cdot \left(\frac{1}{2}\right)^{2n}$

$$= \frac{(2n)!}{(n!)^2} \cdot \frac{1}{2^{2n}} \approx ?$$

Stirling's formula: $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$ will show: $c\sqrt{n}\, n^n e^{-n}$

Then:

$$P_{2n}(0,0) \approx \frac{c\sqrt{2n}\,(\cancel{2n})^{2n}\,\cancel{e^{-2n}}}{c^2\, n\, \cancel{n^{2n}}\, \cancel{e^{-2n}}} \cdot \frac{1}{\cancel{2^{2n}}} = \frac{\sqrt{2}}{c\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

$$\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \dots$$

$$= 1 + \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} + 8 \times \frac{1}{16} + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty$$

So $\sum P_{2n}(0,0) = \infty \implies$ RW is <u>recurrent</u>

Hand waving:

$\sigma = \underbrace{E[X_1^2]}_{1} - \underbrace{E[X]^2}_{0} = 1$

CLT $\frac{X_n}{\sigma\sqrt{n}} \sim N(0,1)$

So $X_n \sim N(0,n)$

"say" $X_n \sim \text{unif}(-\sqrt{n}, \sqrt{n})$. Then $P_{2n}(0,0) \sim \frac{c}{\sqrt{n}}$

$d=3$: Guess? $P_{2n}(0,0) \sim \left(\frac{c}{\sqrt{n}}\right)^3 = \frac{c}{n^{3/2}}$

So $\sum P_{2n}(0,0) < \infty$ transient?