Lec 13

Linear difference eqns.

$$f(n) = a f(n-1) + b f(n+1) \qquad k < n < l$$

$$f(k) = c_0 \qquad f(k+1) = c_1$$

$$f(n+1) = \frac{1}{b}\left(f(n) - a f(n-1)\right)$$ says $f(n)$ should grow exponentially

So $f(k), f(k+1) \to f(k+2) \to \cdots \to f(n)$

What about $f(k) = c_0$ and $f(l) = c_1$?

Guess: $f(n) = u^n$?

$$u^n = a u^{n-1} + b u^{n+1}$$

$$u = a + b u^2 \qquad b u^2 - u + a = 0$$

$$u = \frac{1 \pm \sqrt{1 - 4ab}}{2b}$$

$4ab \neq 1 \Rightarrow$ two solutions u_1 & u_2 (could be complex)

Then $f(n) = \alpha u_1^n + \beta u_2^n$

plug in $n = k$ & l ?

$$\left.\begin{aligned} c_0 &= \alpha u_1^k + \beta u_2^k \\ c_1 &= \alpha u_1^l + \beta u_2^l \end{aligned}\right)$$ solve for α & β

$4ab = 0 \Rightarrow$ one solution $u = \frac{1}{2b}$

Turns out in this case nu^n also works!

So $f(n) = \alpha u^n + \beta n u^n$

again, plug in $n = k$ & l and solve for α & β

e.g. Fibonacci $a_{n+1} = a_n + a_{n-1}$ $\quad a_0 = 1$, $a_1 = 1$

$u^{n+1} = u^n + u^{n-1} \longrightarrow u^2 = u + 1 \rightarrow \frac{1 \pm \sqrt{5}}{2}$

So $a_n = \alpha \left(\frac{1-\sqrt{5}}{2}\right)^n + \beta \left(\frac{1+\sqrt{5}}{2}\right)^n$

$1 = \alpha + \beta$ $\quad 1 = \alpha \left(\frac{1-\sqrt{5}}{2}\right) + \beta \left(\frac{1+\sqrt{5}}{2}\right)$

$(n=0)$ $\quad (n=1)$

$$= \frac{\alpha + \beta}{2} + \frac{\sqrt{5}}{2}(\beta - \alpha)$$

$\beta - \alpha = \frac{1}{\sqrt{5}}$

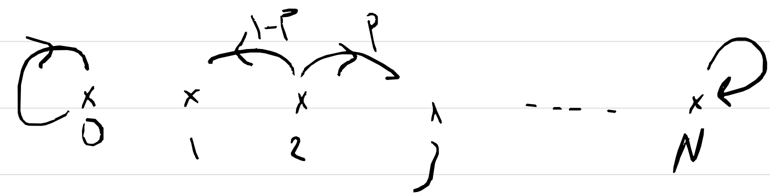
$\beta + \alpha = 1$ $\quad \beta = \frac{1+\sqrt{5}}{2\sqrt{5}}$ $\quad \alpha = \frac{\sqrt{5}-1}{2\sqrt{5}}$

$$a_n = \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right]$$

always an integer!!

Gambler's ruin



$P(\text{reach } N \mid X_0 = i) = ?$ $i = 1, \dots, N-1$

Can't use MS b/c N is unknown!

Let $a(j) = P(\text{reach } N \mid X_0 = j)$

$a(0) = 0$ $a(N) = 1$

$a(j) = (1-p)\, a(j-1) + p\, a(j+1)$

Difference eqn

$u^j = (1-p)\, u^{j-1} + p\, u^{j+1}$

$p u^2 - u + (1-p) = 0$ $1 - 4p(1-p) = 1 - 4p + 4p^2 = (1-2p)^2$

$p \neq \frac{1}{2}$: $\dfrac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \dfrac{1 \pm (1-2p)}{2p} = \begin{cases} 1 \\ \frac{1-p}{p} \end{cases}$

so $a(j) = \alpha + \beta \left(\frac{1-p}{p}\right)^j$

$\left.\begin{array}{l} 0 = a(0) = \alpha + \beta \quad \text{so} \quad \beta = -\alpha \\ 1 = a(N) = \alpha\left(1 - \left(\frac{1-p}{p}\right)^N\right) \end{array}\right\} \Rightarrow$

$$\boxed{a(j) = \frac{1-\left(\frac{1-p}{p}\right)^j}{1-\left(\frac{1-p}{p}\right)^N}}$$ (note: always >0)

$p=\frac{1}{2}$: One solution $u=1$

so $a(j) = \alpha + \beta j$

$0 = a(0) = \alpha$ so $\alpha = 0$

$1 = a(N) = \beta N$ so $\beta = \frac{1}{N}$

$$\boxed{a(j) = \frac{j}{N}}$$

$$0 \xleftrightarrow[j]{} j \xleftrightarrow[N-j]{} N$$

Lec14

In this case, let T be # steps till walk reaches 0 or N

$G(j) = E[T \mid X_0 = j] = ?$

Could use Q^{-1} but N unknown.

$G(0) = 0 \qquad G(N) = 0$

$$G(j) = \frac{1}{2}\left(G(j+1)+1\right) + \frac{1}{2}\left(G(j-1)+1\right)$$

$$= \frac{1}{2}G(j+1) + \frac{1}{2}G(j-1) + 1$$

messes up linear differences