Lec 11

Suppose i & j are in same <u>rec.</u> class

$E[\text{time to get to } j \mid X_0 = i] = ?$

$j = i \Rightarrow 0$

$j \neq i$?

$$\text{Rec class of } j \quad \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

← focus on smaller trans. matrix

$$\begin{array}{c} j \\ \text{rest} \end{array} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & \text{as is} & & \end{bmatrix}$$

change transitions <u>from</u> j to make it <u>absorbing</u>

Now, the <u>only</u> rec. class is $\{j\}$!

$i \neq j$: $E[\text{time to get to } j \mid X_0 = i]$

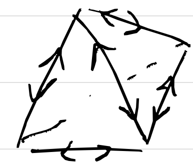
$= E[\text{total \# of visits to all other states} \mid X_0 = i]$

$= \sum$ of the i row in $M = (I - Q)^{-1}$!

$$\begin{bmatrix} 1 & 0 \cdots 0 \\ [\;] & Q \end{bmatrix}$$

Ex.

$$\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$



$$E_1[\text{time to get to } 4] = ?$$

$$\begin{array}{c} \\ 4 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 4 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

Q

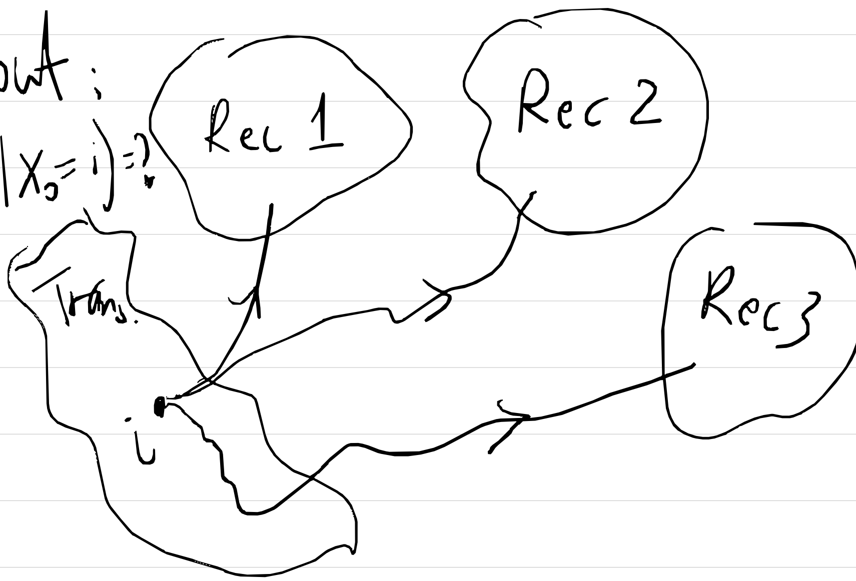
$$I - Q = \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix}$$

$$(I-Q)^{-1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3/2 & \frac{3}{4} & \frac{3}{4} \\ 3/4 & 3/2 & 3/4 \\ 3/4 & 3/4 & 3/2 \end{bmatrix} \end{array}$$

$$E_1[\tau_4] = \frac{3}{2} + \frac{3}{4} + \frac{3}{4} = 3$$

What about:

P(end up in Rec 1 | $X_0 = i$) = ?



$$P = \begin{array}{r} \text{Rec1} \\ \text{Rec2} \\ \vdots \\ \text{trans} \end{array} \begin{bmatrix} \boxed{} & 0 & 0 \\ 0 & \boxed{} & 0 \\ 0 & 0 & \ddots \\ \multicolumn{3}{c}{\boxed{}} \end{bmatrix}$$

Collapse the recurrent classes to single states. Also, ignore the other recurrent classes.

$$\begin{array}{r|c} & \begin{array}{cccc} ① & ② & \cdots & \text{trans.} \end{array} \\ \begin{array}{r} \text{collapsed rec. classes} \left\{ \begin{array}{c} ① \\ ② \\ \vdots \end{array} \right. \\ \text{trans. states} \{ \end{array} & \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & 0 & 0 \\ & 0 & \ddots & \\ \multicolumn{3}{c}{\boxed{}} & Q \end{bmatrix} \end{array}$$

transient-to-transient probabilities remain the same

the new transition probabilities are the sum of transitions to members of the collapsed class

This is what we called Q

identity matrix

zeros

So new $P = \begin{bmatrix} I & 0 \\ S & Q \end{bmatrix}$

collapsed probabilities

Let $r_1, \ldots, r_m$ be the rec. <u>absorbing</u> states
$t_1, \ldots, t_\ell$ the transient states

Let $\alpha(t_i, r_j) = P(\text{absorbed} @ r_j \mid X_0 = t_i)$

Claim $\underset{\ell \times m}{A} = \underset{\ell \times m}{S} + \overset{\ell \times \ell}{Q}\,\underset{\ell \times m}{A}$

so $A - QA = S$, $(I-Q)A = S$

thus $A = (I-Q)^{-1}S = MS$!

Proof.

law of total probability

$$\alpha(t_i, r_j) = \sum_{x \in S} \underbrace{P(X_1 = x \mid X_0 = t_i)}_{P_{t_i,x}} \underbrace{P\left(\begin{smallmatrix}\text{absorbed}\\ \text{at } r_j\end{smallmatrix} \,\middle|\, X_1 = x\right)}_{\alpha(x, r_j)}$$

$$= \sum_{x \text{ absorbing}} P_{t_i,x} \underbrace{\alpha(x, r_j)}_{\substack{1 \text{ if } x = r_j \\ 0 \text{ o/w}}} + \sum_{x \text{ transient}} \overset{Q_{t_i,x}}{P_{t_i,x}} \alpha(x, r_j)$$

$$= \overset{S_{t_i r_j}}{P_{t_i r_j}} + (QA)_{t_i, r_j}$$

$\square$