Lec10

$$E\left[\sum_{k=0}^{\infty} 1\{X_k=j\} \mid X_0=i\right] = ?$$

? avg. ~~total~~ # of visits to j, starting at i

$= \infty$ if $i \to j$ & j rec. (e.g. i rec. j rec. in same class)

(b/c MC has a positive prob. to get to j and when it does, it visits j infinitely often!)

$=0$ if $i \not\to j$ (e.g. i rec. & j trans or i rec. & j rec. in another class)

So the only nontrivial case is: i & j both transient.

$$? = \sum_{k=0}^{\infty} P(X_k=j \mid X_0=i) = \sum_{k=0}^{\infty} (P^k)_{ij}$$

$$= \left(\sum_{k=0}^{\infty} P^k\right)_{ij}$$

recall $1+x+x^2+\dots = \frac{1}{1-x}$ works for matrices too!

so $= (I-P)^{-1}$

but $\det(I-P)=0$ b/c P has e-val 1 !

$$P = \begin{array}{c|cc} & \text{Rec} & \text{Trans} \\ \hline \text{Rec} & \tilde{P} & 0 \\ \text{Trans} & S & Q \end{array} \qquad P^k = \begin{bmatrix} \tilde{P}^k & 0 \\ \text{junk} & Q^k \end{bmatrix}$$

i & j trans. $\Longrightarrow$ we only care about Q^k:

$$\left(\sum_{k=0}^{\infty} Q^k\right)_{ij} = (I-Q)^{-1}_{ij}$$

Q only has e-val w/ $|\cdot|<1$
so $I-Q$ is invertible!

$Q^k \to 0$ b/c $P(X_k=j \mid X_0=i) \to 0$ as $k \to \infty$

Let $\boxed{M = (I-Q)^{-1}}$

$$\boxed{E[\#\text{ visits to } j \mid X_0 = i] = M_{ij}}$$

careful: # visits to j
so if $j=i$, $X_0=i$ counts as a <u>visit</u>
so M's diagonal entries are always ≥ 1

Example:

$$P = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\{1\}$ $\{5\}$ rec.

$\{2,3,4\}$ trans.

Rewrite

$$\begin{array}{c} 1 \\ 5 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$\uparrow Q$

$$I - Q = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$M = \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 3/2 & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{bmatrix}$$

Note diagonals ≥ 1

$X_0 = 2$: $E_2[\#\text{ visits to } 4] = \frac{1}{2}$

note: doesn't have to be an integer b/c it's an avg.

$E_2[\#\text{ steps before absorption into } \{1\} \text{ or } \{5\}]$

$= E_2[\#\text{visits to } 2,3,4] = \frac{3}{2} + 1 + \frac{1}{2} = 3$