Lec 9

Proof of formula: in a recurrent class

Fix i and consider the MC starts there.

Let $r(j)$ = avg. # visits to j before returning to i.

$$= E_i\left[\sum_{k=0}^{\tau_i - 1} 1\{X_k = j\}\right]$$

Formula for $r(j)$?

$r(i)=1$ (the only visit to i before returning to it is at time $n=0$)

what about $j \neq i$?

Claim: $rP = r$!

If so, then (when we have a single rec. class): $r = a\,\Phi_\infty$

Since $r(i)=1$, we get $a = \dfrac{1}{\Phi_\infty(i)}$

So $r(j) = \dfrac{\Phi_\infty(j)}{\Phi_\infty(i)}$

add over j: $\sum_{j \in S} r(j) = \dfrac{1}{\Phi_\infty(i)}$ also $= E_i[\tau_i]$!

proof of claim:

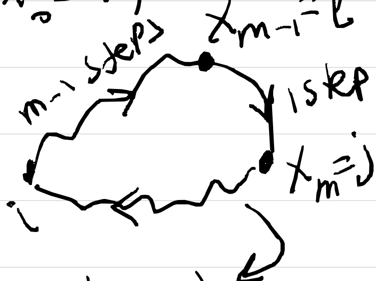
$$rP(j) = \sum_{\ell \in S} r(\ell) P_{\ell,j}$$

$$= \sum_{\ell \in S} P_{\ell,j} \, E_i\Big[\sum_{k=0}^{\tau_i - 1} 1\{X_k = \ell\}\Big]$$

$$= \sum_{\ell \in S} P_{\ell,j} \, E_i\Big[\sum_{k=0}^{\infty} 1\{X_k = \ell, k \le \tau_i - 1\}\Big]$$

$$= \sum_{\ell \in S} P_{\ell,j} \sum_{k=0}^{\infty} P(X_k = \ell, k+1 \le \tau_i \mid X_0 = i)$$

$$= \sum_{\ell \in S} P_{\ell j} \sum_{m=1}^{\infty} P(X_{m-1} = \ell, m \le \tau_i \mid X_0 = i)$$



$$= \sum_{m=1}^{\infty} \underbrace{\sum_{\ell \in S} \underbrace{P(X_{m-1} = \ell, m \le \tau_i \mid X_0 = i) P_{\ell j}}_{P(X_{m-1}=\ell,\, X_m = j,\, m \le \tau_i \mid X_0 = i)}}_{P(X_m = j,\, m \le \tau_i \mid X_0 = i)}$$

Markov property

$P(X_m = j, m \le \tau_i \mid X_0 = i)$ ← law of total probab.

$$= E_i\Big[\sum_{m=1}^{\tau_i} 1\{X_m = j\}\Big]$$

when counting # visits, can replace first by last!

(!) $$= E_i\Big[\sum_{m=0}^{\tau_i - 1} 1\{X_m = j\}\Big] = r(j)$$