Lec 8

Suppose we have one rec. class
(possibly w/ period > 1)

Define:

$$T_{ij}(n) = E\Big[\underbrace{\sum_{k=0}^{n} 1\{X_k = j\}}_{} \,\Big|\, X_0 = i\Big]$$

↑ # visits to j by time n

$$\frac{1}{n} T_{ij}(n) = \frac{1}{n} \sum_{i=0}^{n} \underbrace{P(X_k = j \mid X_0 = i)}_{\phi_k(j)}$$

(w/ $\phi_0 = [0 \ldots 1 0 \ldots]$ at i)

one block of size ≤ d

about $\frac{n}{d}$ blocks of size d

$\rightarrow \Phi_\infty(j)$ by Perron-Frobenius

$$\frac{[\cdots]/d + [\quad]/d + \cdots + [\phi_{n-d+1}(j) + \cdots + \phi_{n-1}(j) + \phi_n(j)]/d}{n/d} \rightarrow \Phi_\infty(j)$$

So $\boxed{\Phi_\infty(j) = \text{limiting proportion of time spent at } j \text{ [when started at } j\text{'s class]}}$

More generally, the limit is:

$$P(\text{MC gets to } j\text{'s class} \mid X_0 = i) \times \Phi_\infty(j)$$

↑ how to compute?

Define:

$\tau_i = \min\{n \geq 1 : X_n = i\}$

$\uparrow$ matters! if $X_0 = i$, we look for the first <u>return</u> to i

Big Picture: $\begin{cases} 1 \text{ visit takes } \approx E[\tau_i \mid X_0 = i] = E_i[\tau_i] \text{ steps} \\ k \text{ visits } \approx k E_i[\tau_i] \text{ steps} \end{cases}$

n large $\Rightarrow$ n steps should have $\approx n \Phi_\infty(i)$ visits to i

so k large $\rightsquigarrow$ take $n = k E_i[\tau_i]$
we should have $\approx k E_i[\tau_i] \Phi_\infty(i)$ visits

So $k \approx k E_i[\tau_i] \Phi_\infty(i)$

and $\boxed{E_i[\tau_i] = \frac{1}{\Phi_\infty(i)}}$ $\leftarrow$ average time to return to i, starting at i

Now, we prove this!

Note: both sides $= \infty$ if i is in a transient class.