Lec 7

If $d > 1$, we cannot have $P^n \to \begin{bmatrix} \Phi_\infty \\ \vdots \\ \Phi_\infty \end{bmatrix}$ (recall $1 \rightleftarrows 2$)

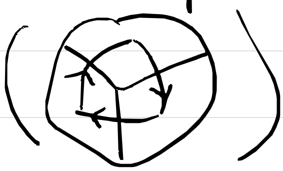
So what happens to Φ_n as $n \to \infty$?

Because of cycles, should look at Φ_n along n w/ fixed remainder in n/d.

So if $d = 3$, we look at 3 sequences:

n = multiples of 3: 0, 3, 6, 9, 12, ...
n = 1 + multiples of 3: 1, 4, 7, 10, 13, ...
n = 2 + multiples of 3: 2, 5, 8, 10, 14, ...

each will have its own limit



Th. $\dfrac{\Phi_n + \Phi_{n+1} + \cdots + \Phi_{n+d-1}}{d} \xrightarrow[n\to\infty]{} \Phi_\infty$

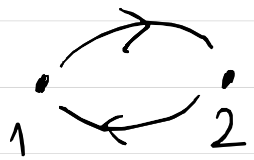
and starting at a site i, when n is large, only certain sites are accessible, depending on the remainder of n/d.

$\Phi_n \longrightarrow d\,[\underbrace{0\cdots0}\ \boxed{\Phi_\infty}\ \underbrace{0\cdots0}\ \boxed{\Phi_\infty}\ \underbrace{0\cdots0}]$

accessible sites

inaccessible sites

Baby example:



$$\begin{array}{cc} 1-a & a \\ \downarrow & \downarrow \end{array}$$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{array}{cc} b & 1-b \end{array}$$

$d = 2 \qquad \Phi_\infty = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

unique

If we start at 1:

$\Phi_n = [1 \;\; 0]$ for n even

and $= [0 \;\; 1]$ for n odd

So $\Phi_n \longrightarrow$ $[1 \;\; 0]$ as $n \to \infty$ along evens

$[0 \;\; 1]$ as $n \to \infty$ along odds

$[1 \;\; 0]$

inaccessible from 1

$d = 2$ times $\Phi_\infty(1) = \frac{1}{2} = 1$

$[0 \;\; 1]$

inaccessible from 1 $\quad d\,\Phi_\infty(2) = 2 \cdot \frac{1}{2} = 1$

Slightly different if we start at 2: even $\leftrightarrow$ odd

Φ_0 general $\Rightarrow$ Φ_n cycles through d limits

but it is messy to sort that out analytically.

In and case: $\dfrac{\Phi_n + \Phi_{n+1} + \cdots + \Phi_{n+d-1}}{d} \to \Phi_\infty$

accessible sites are disjoint!

Idea of Proof.

$\lambda_1 \cdots \lambda_m$ e-values

$v_1 \cdots v_m$ left e-vectors

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} P = \begin{bmatrix} \lambda_1 v_1 \\ \vdots \\ \lambda_m v_m \end{bmatrix} = \overset{D}{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix}} \overset{Q}{\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}}$$

$$P = Q^{-1} D Q \qquad P^n = Q^{-1} \underbrace{D^n}_{\text{easy}} Q$$

rec. class

$d-1$ evals in $\mathbb{C}$ w/ norm 1

$$D = \begin{bmatrix} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & \ddots \end{matrix} & 0 & \\ 0 & \boxed{} & \\ & & \ddots \end{bmatrix}$$

another rec. class

transients: e-val in $\mathbb{C}$ but with norm <1