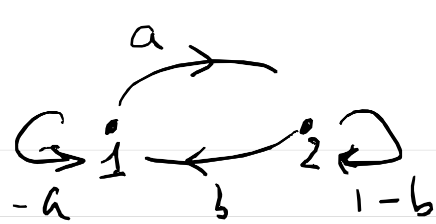
$$\bar{\phi}_\infty = \left[\frac{b}{a+b} \quad \frac{a}{a+b}\right]$$



$\uparrow \uparrow$ intuitive!

"after a long time, spend at 1 & 2 a time proportional to b & a"

unique if $a > 0$ or $b > 0$
(o/w G G !)

Lec 4

Alternative ordering of steps:

$$\bar{\phi}_\infty P = \bar{\phi}_\infty \Rightarrow \bar{\phi}_\infty (P - I) = [0 \cdots 0]$$

One redundant eqn. (b/c P stochastic)
so replace on column in $P - I$ w/ all 1.

And replace $[0 \cdots 0]$ w/ $[0 \cdots 0 \; 1 \; 0 \cdots 0]$ location of changed column

(New equation: sum of entries = 1)

Now: $\bar{\phi}_\infty = [0 \cdots 0 \, 1 \, 0 \cdots 0] \left(\text{new } P - I\right)^{-1}$

$= i^{\text{th}}$ row of $(\text{new } P - I)^{-1}$

if we replaced column #i

$$\begin{array}{ccccc} P & & P-I & & \text{new } P-I \\ \begin{bmatrix}1-a & a\\ b & 1-b\end{bmatrix} & \to & \begin{bmatrix}-a & a\\ b & -b\end{bmatrix} & \to & \begin{bmatrix}-a & 1\\ b & 1\end{bmatrix} \end{array}$$

replaced last column

inverse

$$\to \begin{bmatrix}-a & 1\\ b & 1\end{bmatrix}^{-1} = \begin{bmatrix}-\frac{1}{a+b} & \frac{1}{a+b}\\ \frac{b}{a+b} & \frac{a}{a+b}\end{bmatrix}$$

$$\to \Phi_\infty = \begin{bmatrix}\frac{b}{a+b} & \frac{a}{a+b}\end{bmatrix} \quad \checkmark$$

$\leftarrow$ last row of

Back to Φ_n as $n \to \infty$.

$\Phi_\infty P = \Phi_\infty$ gives "candidates" for the limit

But does the convergence hold?

When Φ_∞ is unique, it is reasonable to expect $\Phi_n \to \Phi_\infty$ (will see later)

Then $P(X_n = i) \approx \Phi_\infty(i)$

Regardless of the value of Φ_0 !!

So $\Phi_0 P^n \to \Phi_\infty$

$\Phi_0 = [0 \cdots 0\ 1\ 0 \cdots 0] \rightsquigarrow$ all rows of P^n $\longrightarrow \Phi_\infty$

$$P^n \to \begin{bmatrix} \bar{\Phi}_\infty \\ \bar{\Phi}_\infty \\ \vdots \\ \bar{\Phi}_\infty \end{bmatrix}$$ ← stochastic matrix

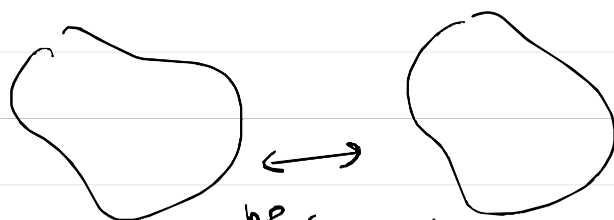
&

$$\bar{\Phi}_0 \begin{bmatrix} \bar{\Phi}_\infty \\ \vdots \\ \bar{\Phi}_\infty \end{bmatrix} = \bar{\Phi}_\infty \text{ b/c } \sum_{i \in S} \bar{\Phi}_0(i) = 1$$

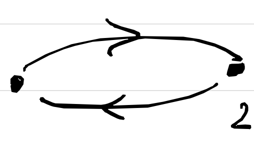
But when DOES $\bar{\Phi}_n \to \bar{\Phi}_\infty$?

When IS $\bar{\Phi}_\infty$ unique

What issues can we have?

1) 

⇒ multiple invariant measures

2) 

$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$!

So $P^n \not\to$ depends on $n <$ even / odd

Note if P has positive entries, all is good

<u>Th.</u> P has positive entries ⇒ $\bar{\Phi}_\infty$ unique and $\bar{\Phi}_n \to \bar{\Phi}_\infty$