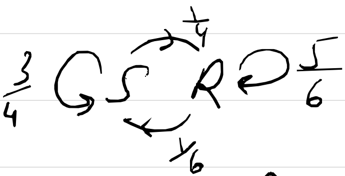
Lec 3

S = sunny R = rainy

E.g.

$$P = \begin{array}{c c} & \begin{matrix} S & R \end{matrix} \\ \begin{matrix} S \\ R \end{matrix} & \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix} \end{array}$$

$\bar{\phi}_0 = [1\ 0]$ (starting "sunny")



$P(X_6 = S) = ?$

$$P^6 = \begin{array}{c c} & \begin{matrix} S & R \end{matrix} \\ \begin{matrix} S \\ R \end{matrix} & \begin{bmatrix} 0.424 & 0.576 \\ 0.384 & 0.616 \end{bmatrix} \end{array}$$

$P(X_1 = R, X_2 = S, X_8 = R) = ?$

b/c $X_0 = S \longrightarrow P_{SR}\ P_{RS}\ (P^6)_S^R = \frac{1}{4} \cdot \frac{1}{6} \cdot 0.576$

If instead $\bar{\phi}_0 = [\frac{1}{3}\ \frac{2}{3}]$ then: $(\phi_0 P)_R$

$P(X_1 = R, X_2 = S, X_8 = R) = P(X_1 = R) \cdot P_{RS}\ (P^6)_{SR}$

$$\phi_0 P = [\tfrac{1}{3}\ \tfrac{2}{3}] \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix} = [\underbrace{\tfrac{1}{4} + \tfrac{1}{9}}_{S} \quad \underbrace{\tfrac{1}{12} + \tfrac{5}{9}}_{R}]$$

$P(X_1 = R) = \frac{1}{12} + \frac{5}{9}$.

What happens after a long time? $n \to \infty$

$\phi_n = \phi_0 P^n$ so P^n $n \to \infty$?

P^n easy to calculate if P is diagonal

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}^n = \begin{bmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_k^n \end{bmatrix}$$

Later: will use eigenvalues & eigenvectors

$\phi_n = \phi_{n-1} P \qquad n \to \infty \qquad \phi_n \to \phi_\infty$

means $\phi_\infty = \phi_\infty P$

invariant measure

(start w/ $\phi_\infty \rightsquigarrow$ always at ϕ_∞)

same

$Pu = \lambda u \qquad\qquad vP = \lambda v$

e-vetor

e-value

e-val

e-vect

<u>right</u> <u>left</u>

We care about $\lambda = 1$

$P \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ b/c rows add up to 1

So 1 is a right e-value
hence also a left e-value

So $\exists\ \phi_\infty$ not identically 0 s.t. $\phi_\infty = \phi_\infty P$

but are entries ≥ 0 ?

Then ϕ_∞ is <u>normalized</u> to add up to 1

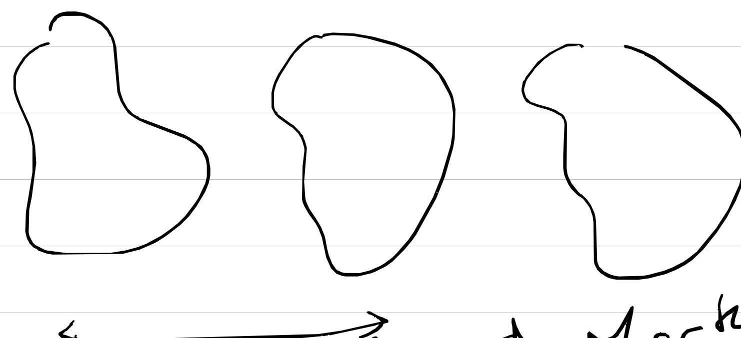
<u>Th.</u> (Perron-Frobenius) Yes! when P is stochastic

$\bar{\Phi}_\infty$ is then normalized to add up to 1 (i.e. divide by sum of entries)

⚠ There may be more than one Perron-Frobeinus e-vect.!

$\rightarrow$ more than one inv. meas.!

e.g.



three different Markov chains that do not communicate, but are viewed as a single MC!

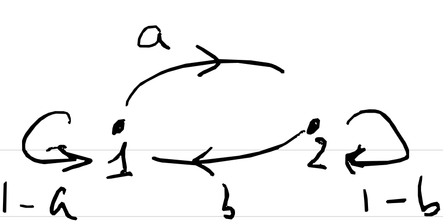
<u>E.g.</u> $P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$

$\bar{\Phi}_\infty = [x \;\; y]$ $\quad \bar{\Phi}_\infty P = \bar{\Phi}_\infty$: $\begin{aligned} (1-a)x + by &= x \\ ax + (1-b)y &= y \end{aligned}$

so $ax = by$ (reduced to one equation b/c P is stochastic!)

$\underline{\underline{x + y = 1}}$ $\quad \boxed{x = \frac{b}{a+b} \quad y = \frac{a}{a+b}}$

$$\bar{\Phi}_\infty = \left[\frac{b}{a+b} \quad \frac{a}{a+b}\right]$$



intuitive!

"after a long time, spend at 1 & 2 a time proportional to b & a"

unique if $a>0$ or $b>0$ (o/w G G !)

Lec4

Alternative ordering of steps:

$$\bar{\Phi}_\infty P = \bar{\Phi}_\infty \Rightarrow \bar{\Phi}_\infty (P-I) = [0 \cdots 0]$$

One redundant eqn. (b/c P stochastic) so replace on column in $P-I$ w/ all 1.

And replace $[0 \cdots 0]$ w/ $[0 \cdots 0\ 1\ 0 \cdots 0]$ location of changed column

(New equation: sum of entries $=1$)

Now: $\bar{\Phi}_\infty = [0 \cdots 0\ 1\ 0 \cdots 0]\left(\text{new } P-I\right)^{-1}$

$= i^{th}$ row of $(\text{new } P-I)^{-1}$

if we replaced column $\# i$